DOI 10.15622/sp.2019.18.6.1407-1433

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APPLICATION OF DIFFERENCE SCHEMES TO DECISION THE PURSUIT PROBLEM

Ochkov V.F., Vasileva I.E. **Application of Difference Schemes to Decision the Pursuit Problem.**

Abstract. The problem of the pursuit curve construction in the case when the tangent to pursuer's motion trajectory passes at any time through the point representing the pursued is considered. A new approach to construct the pursuit curves using difference schemes is proposed. The proposed technique eliminates the need to derive the differential equations for the description of the pursuit curves, which is quite difficult task in the general case. In addition, the application of difference methods is justified in a situation where it is complicated to find the analytical solution of an existing differential equation and it is possible to obtain the pursuit curve only numerically. Various modifications of difference schemes respectively equivalent to the Euler, to the Adams – Bashforth and to the Milne methods are constructed. Their software implementation is realized by using the mathematical package Mathcad. We consider the case of a uniform rectilinear motion of the pursued whose differential equation describing the path of the pursuer and its analytical solution are known. We compare the numerical solutions obtained by the different methods with the well-known analytical solution. The error of the obtained numerical solutions is examined. Moreover, an application is considered illustrating the construction of the difference schemes for the case of an arbitrary trajectory of the pursued. Also, we extend the proposed method to the case of cyclic pursuit with several participants in the three-dimensional space. In particular, we construct a difference scheme equivalent to the Euler method for a three-dimensional analogue of the "bugs problem". The results obtained are demonstrated by means of animated examples for either two-dimensional or three-dimensional cases.

Keywords: Differential Games, Pursuit Problem, Pursuit Curve, Numerical Methods, Difference Methods, the Euler Method, "Three Bugs" or "Three Mice" Problem, Mathcad.

1. Introduction. The pursuit problem belongs to a class of long and widely studied problems. The leading role in the formulation of this problem belongs to research conducted in the field of the differential games, where it is necessary to choose the optimal pursuit strategy [1-6]. A common strategy is the pursuit method [7] which determines the motion of the pursuer in such way that the tangent to the trajectory of its motion at any time passes through the position of the point associated with the pursued. The problem of constructing the trajectory of the pursuer — the pursuit curve — is relevant in various areas and it has a wide practical value, in particular, in mechanics, military affairs, control systems [8-25].

Despite the variety of applied problems, only particular cases of pursuit are sufficiently studied, for example, the case of a simple motion when the pursued entity moves uniformly along a straight line. For this case, it is possible to explicitly write a second-order nonlinear differential equation describing the pursuit curve and to find its analytical solution [9, 10]. It is assumed, for definiteness, that the pursued entity begins a uniform motion with speed $\bar{v}$ along the axis Oy from the origin of coordinates, while the pursuer starts with speed $\bar{V}$ from the point with coordinates (1,0) (Figure 1).

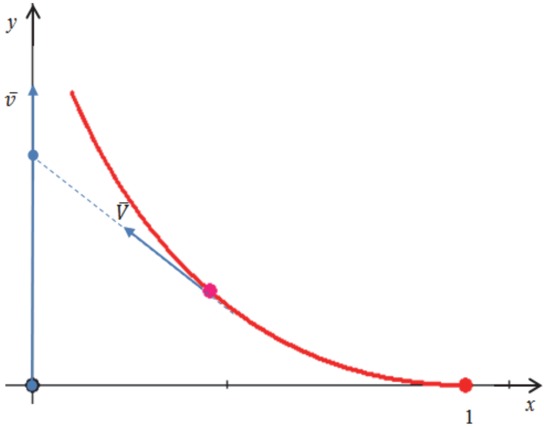


Fig. 1. Scheme of a simple motion

Then the sought differential equation is given by [9, 10]:

$$y''(x) = \frac{1}{kx} \cdot \sqrt{1 + (y'(x))^2}, \quad (1)$$

where $k = v / V$ is the ratio of the speeds of the pursued and of the pursuer.

Taking into account the zero initial conditions $y(1) = 0, y'(1) = 0$, which emerge from the statement of the problem, the solution of equation (1) is described as follows [10]:

$$y(x) = \frac{1}{2}\left(\frac{x^{1+k}}{1+k} - \frac{x^{1-k}}{1-k}\right) + \frac{k}{1-k^2}, \text{if } k \neq 1;$$
$$y(x) = \frac{1}{4}(x^2 - \ln x^2 - 1), \text{if } k = 1. \quad (2)$$

To find the pursuit curve in particular cases, either kinematics methods [11-14] or parametrization methods [8] are also used.

However, the question of describing the curve of the pursuer in the general case remains open due to the complexity of deriving the differential equation itself, as well as finding its analytical solution.

This article proposes a numerical approach for constructing the pursuit curve based on difference schemes. Numerical methods are often used in differential games problems, also in relation to the pursuit curve problem. In particular, the grid method and its variations are widely known [22-24], they are designed to derive the value function in a time-optimal game and the optimal trajectory [22]. This method is also successfully applied to solve the pursuit problem in distributed control systems [25]. The difference schemes investigated in the mentioned works are constructed for an available differential equation describing the pursuit process.

The proposed approach instead makes it possible to abandon the need to derive the differential equation that describes the pursuit trajectory. In addition, the application of this approach is justified in a situation where it is difficult to find the analytical solution of the existing differential equation and it is possible to obtain the pursuit curve only by numerical methods. The possibility of this approach is described, in particular, in reference [26].

2. Construction of a difference scheme equivalent to the Euler method. Initially, it is constructed a difference scheme for the case of a simple motion (Figure 1) in order to show the approximation of a well-known analytical solution (2).

Then we deal with the parametrization with respect to time t and denote at each time point the known coordinates of the pursued with $x(t), y(t)$ and the unknown coordinates of the pursuer with $X(t), Y(t)$ [8].

According to the law of the uniform rectilinear motion, the coordinates $x(t), y(t)$ of the pursued along the axis Oy, are determined by the formulas:

$$\begin{aligned} x(t) &= 0, \\ y(t) &= v \cdot t. \end{aligned} \tag{3}$$

The pursuit time is divided into n time intervals and the time step is denoted Δt , and the approximate coordinates of the pursued and of the pursuer at each step: x_i, y_i and $X_i, Y_i, i = \overline{0, n}$, respectively. At the initial time, the coordinates have the following values:

$$\begin{aligned} x_0 = 0, \quad y_0 = 0, \\ X_0 = 0, \quad Y_0 = 0. \end{aligned} \tag{4}$$

From (3) it follows that the coordinates of the pursued $x_{i+1}, y_{i+1}, i = \overline{0, n-1}$, are defined as follows:

$$\begin{aligned} x_{i+1} &= 0, \\ y_{i+1} &= y_i + v \cdot \Delta t, \end{aligned} \quad i = \overline{0, n-1}. \tag{5}$$

In order to determine the coordinates of the pursuer at step $i+1$, geometric constructions are used (Figure 2).

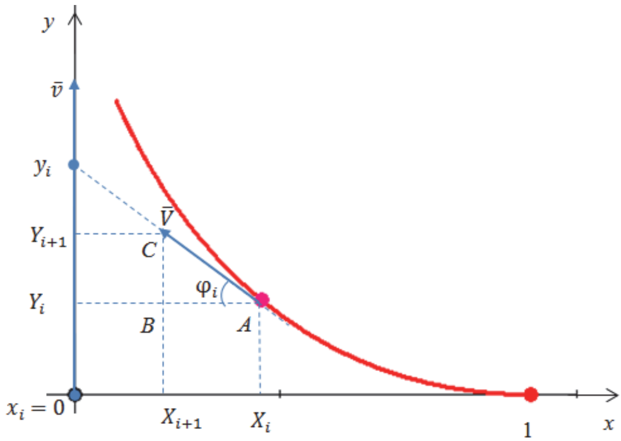


Fig. 2. Construction of the difference scheme

The acute angle between the tangent to the pursuit curve and the *Ox* axis at step i is denoted by φ_i . Obviously, at the initial time $\varphi_0 = 0$. The tangent of the right angle (Figure 2) is given by:

$$\mathrm{tg}(\varphi_i) = \frac{y_i - Y_i}{|x_i - X_i|}, \; i = \overline{0, n-1}. \tag{6}$$

From the right triangle *ABC* we obtain:

$$\begin{aligned} X_{i+1} - X_i &= -AC \cdot \cos\varphi_i, \\ Y_{i+1} - Y_i &= AC \cdot \sin\varphi_i, \end{aligned} \quad i = \overline{0, n-1}, \tag{7}$$

or

$$\begin{aligned} X_{i+1} &= X_i + AC \cdot \cos(\pi + \varphi_i), \\ Y_{i+1} &= Y_i + AC \cdot \sin(\pi + \varphi_i), \end{aligned} \quad i = \overline{0, n-1}. \tag{8}$$

Since the velocity vector $\overline{V}$ is directed tangentially, the following relation applies:

$$AC = \left|\overline{V}\right| \cdot \Delta t. \tag{9}$$

Taking into account (6) and generalizing formula (8) for the cases where the angle φ_i is located in any coordinates quarter, the following recurrence formulas are derived for determining the coordinates of the pursuer:

$$\begin{matrix} X_{i+1} = X_i + \left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_i), \\ Y_{i+1} = Y_i + \left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_i), \end{matrix} \quad i = \overline{0, n-1}, \tag{10}$$

where

$$\overline{\varphi}_i = \begin{cases} \operatorname{arctg}\left(\dfrac{y_i - Y_i}{x_i - X_i}\right) + \pi, \text{if } X_i > x_i, \\ \operatorname{arctg}\left(\dfrac{y_i - Y_i}{x_i - X_i}\right), \text{if } X_i < x_i \quad , \\ \dfrac{\pi}{2} \cdot \operatorname{sgn}(y_i - Y_i), \text{if } X_i = x_i, \end{cases} \tag{11}$$

$$i = \overline{0, n}.$$

It should be noted that the calculation by the difference scheme (10) should be stopped if at any step $k \le n$ the equality $x_k = X_k, y_k = Y_k$ is verified. In fact, this means that the pursuer caught up with the pursued, i.e. the pursuit is complete.

Thus, the central element of the proposed method is the calculation at each step i of the angle $\overline{\varphi}_i$ between the tangent to the pursuit curve and the axis Ox. The expressions $\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_i)$ and $\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_i)$ are respectively approximate increments of functions $X(t)$ and $Y(t)$ per step Δt (Figure 2), and, therefore, at a small step, these expressions serve as analogues of the differentials of the functions $X(t)$ and $Y(t)$ at the point t_i. Therefore, it can be said that the difference scheme (10) is equivalent to the classical difference scheme, which is constructed by the Euler method for finding the solution $u(t)$ of a Cauchy problem of the type:

$$u'(t) = f(t),$$
$$u(t_0) = u_0, \tag{12}$$

and it has the form [27]:

$$u_{i+1} = u_i + \mathrm{d}u(t_i), \quad i = \overline{0, n-1}, \tag{13}$$

where t_i is a partition point, u_i are the values of the approximate solution at the partition points, Δt is the step.

The difference scheme (10) is implemented using the mathematical package Mathcad [28, 29] for given values of the number of steps n , step Δt and speeds v and $V (v \neq V)$. A numerical solution of the problem is obtained in terms of the vectors $(X_i, Y_i), i = \overline{1, n}$, that is, we obtain an approximation of the pursuit curve. A comparison with the analytical solution (2) is made. The results are shown in Figure 3a, b (the animation for Figure 3a is posted on an electronic resource https://community.ptc.com/t5/PTCMathcad/OchkovArticleAnimaions/td-p/583048).

From the graphs shown in Figure 3, it is clear that the obtained numerical solution actually implements the pursuit method (the tangent at the selected point of the pursuer’s trajectory passes through the point of the pursued entity — Figure 3b) and approximates the analytical solution quite well Figure 3a.

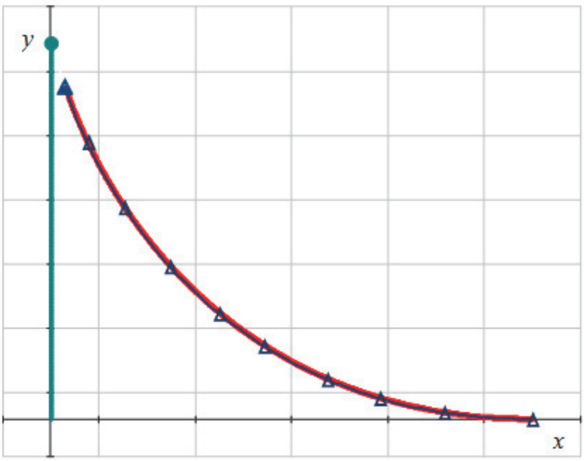


Fig. 3(a). Comparison of numerical and analytical pursuit curves: general view

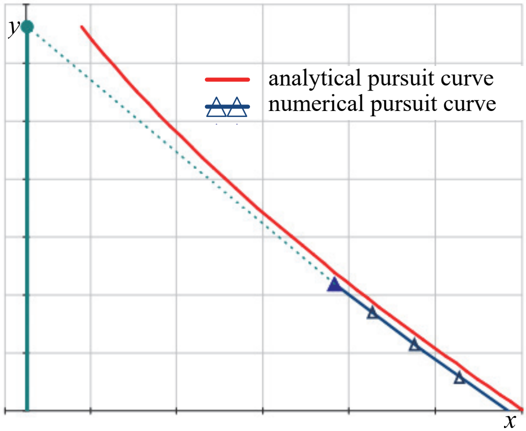


Fig. 3(b). Comparison of numerical and analytical pursuit curves: scaled view of part of the curve

In Figure 4 it is reported a plot showing the error ε of the numerical solution, defined according to the formula:

$$\varepsilon = \left| y\left(X_i\right) - Y_i\left(X_i\right) \right|, \tag{14}$$

where $y(X_i)$ is defined according to (2).

As can be seen from Figure 4, because of accumulation, the error becomes equal to the step $\Delta t = 10^{-3}$, that is, the difference scheme (10) has an accuracy of the first order corresponding to the Euler method [27].

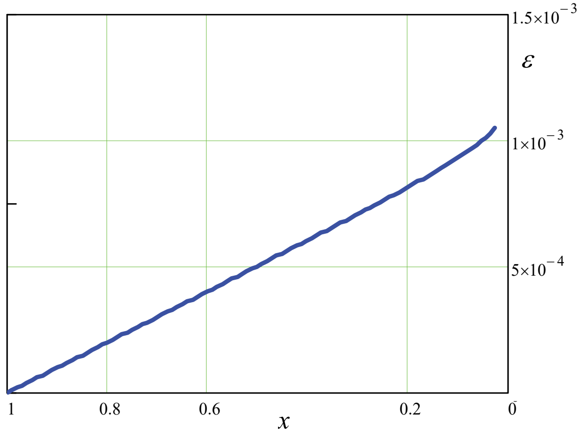


Fig. 4. Numerical solution error

3. Construction of difference schemes using linear multi-step methods. The Euler method, that is used to construct the difference scheme described above, is a rather "rough" method of approximate calculation. To increase accuracy, linear multi-step methods are usually applied. These methods use not only one, but several previously calculated values of the sought function [27]. Let us consider the construction of a difference scheme equivalent to the two-step Adams – Bashforth method, which for a Cauchy problem of the form (12) is given by the formula [27]:

$$\begin{gathered} u_1 = u_0 + \mathrm{d}u(t_0), \\ u_{i+1} = u_i + \frac{3}{2}\mathrm{d}u(t_i) - \frac{1}{2}\mathrm{d}u(t_{i-1}), i = \overline{1, n-1}. \end{gathered} \tag{15}$$

This method has, in contrast to the Euler method, an accuracy of the second order [27].

The difference scheme (10) is modified in accordance with (15). Taking into account the fact that the expressions $\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_i)$ and $\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_i)$ have the meaning of differentials of the functions $X(t)$ and $Y(t)$ at the point t_i, the modified difference method will be:

$$\begin{gathered} X_1 = X_0 + \left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_0), Y_1 = Y_0 + \left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_0), \\ X_{i+1} = X_i + \frac{3}{2}\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_i) - \frac{1}{2}\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i-1}), \\ Y_{i+1} = Y_i + \frac{3}{2}\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_i) - \frac{1}{2}\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i-1}), \\ i = \overline{1, n-1}, \end{gathered} \tag{16}$$

where $\overline{\varphi}_i$ is defined by the formula (11).

The difference scheme (16) is implemented using the Mathcad mathematical package for given values of the number of steps n, step Δt and speeds $v, V (v \neq V)$. The obtained numerical solution is compared with the analytical solution (2) and with the solution obtained using the difference method (10). The results are shown in Figure 5a, b (the animation for Figure 5a is posted on an electronic resource https://community.ptc.com/t5/PTCMathcad/OchkovArticleAnimaions/td-p/583048).

From Figure 5b it is clear that the numerical solution obtained by the difference scheme (16) (equivalent to the Adams – Bashforth method) approximates the analytical solution better than the numerical solution obtained by the difference scheme (10) (equivalent to the Euler method): the plot of this solution coincides visually with the analytical solution.

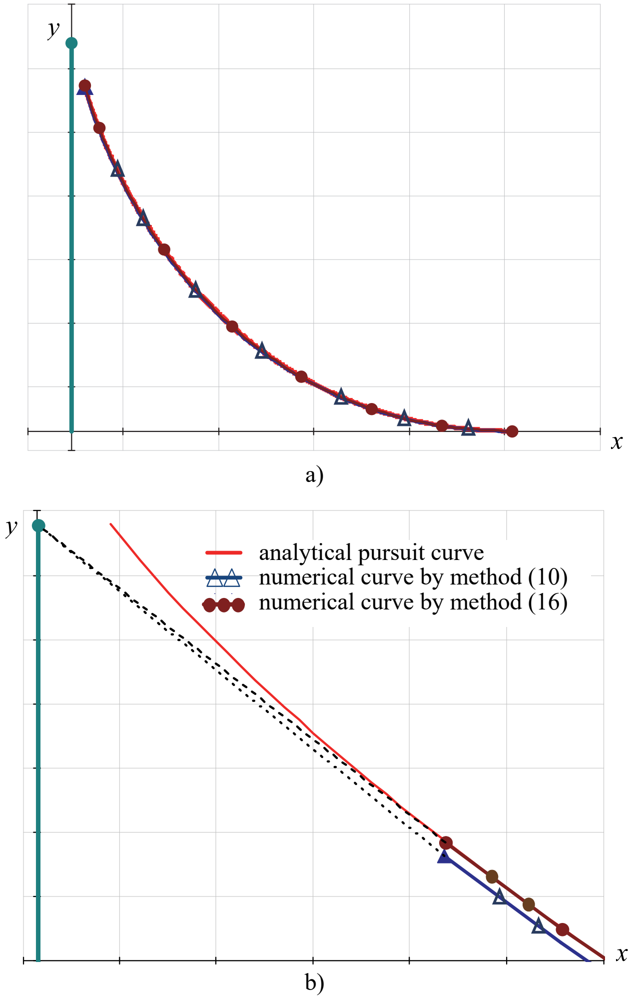


Fig. 5. Comparison of numerical and analytical pursuit curves:
a) general view, b) scaled view of part of the curve

The plot in Figure 6 shows the errors of the numerical solutions, determined by formula (14). The error of the numerical solution obtained by this method (16), for a given step size $\Delta t = 10^{-3}$ is of the order

$O(\Delta t^2)\sim 10^{-6}$, which is far less than the error of the numerical solution obtained by method (10).

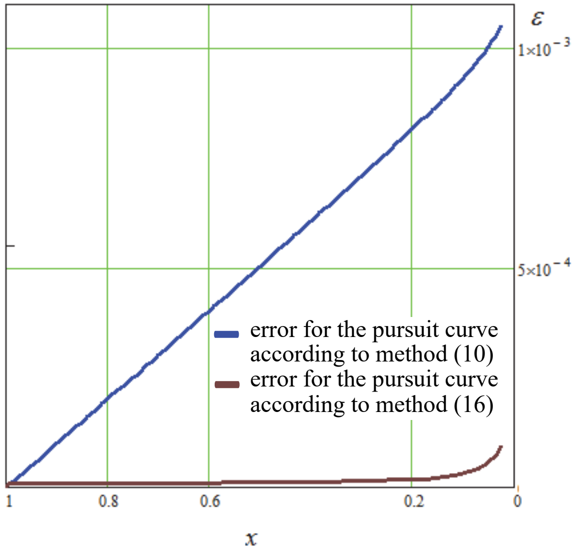


Fig. 6. Numerical error of the solutions obtained by the two methods

If it is necessary to construct the pursuit trajectory with a much smaller error, numerical methods of a higher order should be used, for example, the predictor-corrector methods which are widely used in applications [30-36]. Let us construct the difference scheme equivalent to the fourth-order Milne method [37, 38]. This method requires four initial steps and uses a couple of finite-difference formulas (a predictor and a corrector).

For a given initial value u_0, additional initial values u_1, u_2, u_3 are calculated by some other methods, for example, by the Runge-Kutta method, which has fourth-order accuracy [27, 39, 40].

Let us consider the predictor-corrector formulas by the Milne method starting with the fourth step.

The predictor formula for the Cauchy problem (12) is defined as follows [37]:

$$u_{i+1}^{pred} = u_{i-3} + \frac{4}{3}\left(2\mathrm{d}u\left(t_{i-2}\right) - \mathrm{d}u\left(t_{i-1}\right) + 2\mathrm{d}u\left(t_i\right)\right) \quad (17)$$

$$i = \overline{3, n-1}.$$

The corrector formula is defined as follows [37]:

$$u_{i+1}^{corr} = u_{i-1} + \frac{1}{3}\left(\mathrm{d}u\left(t_{i-1}\right) + 4\mathrm{d}u\left(t_i\right) + \mathrm{d}u^{pred}\left(t_{i+1}\right)\right), \quad i = \overline{3, n-1}, \tag{18}$$

where expression $\mathrm{d}u^{pred}\left(t_{i+1}\right)$ defines a predicted differential of function $u\left(t\right)$ at the point t_{i+1} based on (17).

Let us consider the difference scheme according to (17)-(18).

As noted above, the key element of the proposed method is the calculation at each step *i* of the angle $\overline{\varphi}_i$ between the tangent to the pursuit curve and the axis *Ox*. The expressions $\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_i)$ and $\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_i)$ are respectively analogues of the differentials of the functions $X\left(t\right)$ and $Y\left(t\right)$ at the point t_i.

Accordingly, the essential link of the difference method is the calculation of the predicted angle $\overline{\varphi}_{i+1}^{pred}$.

Thus, the difference method is defined as follows:

$$\begin{gathered}
X_{i+1}^{pred} = X_{i-3} + \frac{4}{3}(2\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i-2}) - \left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i-1}) + \\
+2\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i})), \\
Y_{i+1}^{pred} = Y_{i-3} + \frac{4}{3}(2\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i-2}) - \left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i-1}) + \\
+2\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i})), \\
X_{i+1}^{corr} = X_{i-1} + \frac{1}{3}(\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i-1}) + 4\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i}) + \\
+\left|\overline{V}\right| \cdot \Delta t \cdot \cos(\overline{\varphi}_{i+1}^{pred})), \\
Y_{i+1}^{corr} = Y_{i-1} + \frac{1}{3}(\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i-1}) + 4\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i}) + \\
+\left|\overline{V}\right| \cdot \Delta t \cdot \sin(\overline{\varphi}_{i+1}^{pred})), \\
i = \overline{3, n-1},
\end{gathered} \tag{19}$$

where

$$\overline{\varphi}_{i+1}^{pred} = \begin{cases} \operatorname{arctg}\left(\dfrac{y_{i+1} - Y_{i+1}^{pred}}{x_{i+1} - X_{i+1}^{pred}}\right) + \pi, \text{if } X_{i+1}^{pred} > x_{i+1}, \\ \operatorname{arctg}\left(\dfrac{y_{i+1} - Y_{i+1}^{pred}}{x_{i+1} - X_{i+1}^{pred}}\right), \text{if } \ X_{i+1}^{pred} < x_{i+1}, \\ \dfrac{\pi}{2} \cdot \operatorname{sgn}\left(y_{i+1} - Y_{i+1}^{pred}\right), \text{if } \ X_{i+1}^{pred} = x_{i+1}, \end{cases} \quad (20)$$

$$i = \overline{3, n-1}.$$

The difference scheme (19) is implemented using the Mathcad mathematical package for given values of the number of steps n, step Δt and speeds $v, V (v \neq V)$. The obtained numerical solution is compared with the analytical solution (2) and with the solutions obtained using the difference methods (10), (16). The general view of the plot of the analytical solution and of the numerical solution obtained using the difference methods (19) is identical to the ones in Figures 3, 5.

The plot in Figure 7 shows the errors of the numerical solutions determined according to formula (14) for each of the three solutions obtained by the difference scheme (10), (16), (19).

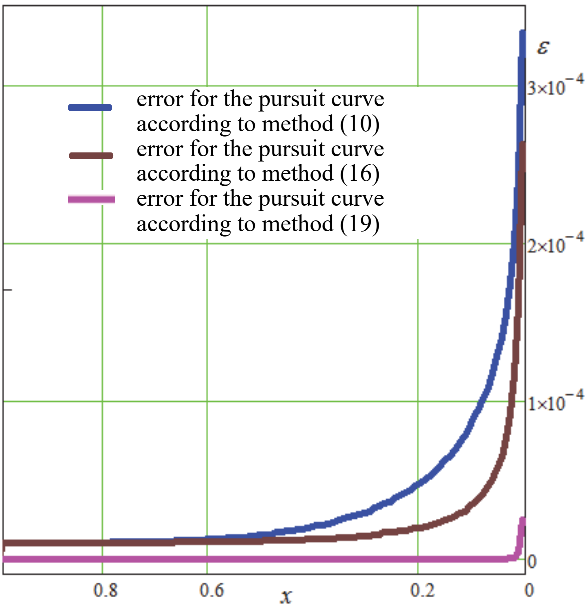


Fig. 7. Numerical error of the solutions obtained by the three methods

The error of the numerical solution obtained by method (19) (equivalent to the Milne method), for a given step size $\Delta t = 10^{-3}$, is of the order $O(\Delta t^4) \sim 10^{-12}$, which is significantly less than the errors of the numerical solutions obtained by methods (10) and (16) (analogue to the Euler method and the Adams-Bashforth method respectively).

A high accuracy of a numerical solution is achieved by implementing of the proposed method.

Therefore, it can be concluded that the proposed difference schemes (10), (16) and (19) allow to construct numerical solutions that approximate the analytical pursuit curve with varying accuracy according to the specific problem.

4. Application of difference schemes for constructing the pursuit curve in case of an arbitrary trajectory of the pursued. The derived difference schemes (10), (16), (19) can be used to obtain a numerical solution also if the pursued entity moves not along a straight line, as it was considered previously, but along an arbitrary trajectory.

Let us consider as an example the situation when the pursued moves uniformly along an elliptical path, i.e. the coordinates $x(t), y(t)$ are determined by the formulas:

$$\begin{aligned} x(t) &= A \cdot \sin(v \cdot t), \\ y(t) &= B \cdot \cos(v \cdot t), \end{aligned} \tag{21}$$

where v is the speed of the pursued, while A, B are constant coefficients.

Let the pursued start moving from the point with coordinates:

$$x_0 = 0,\ y_0 = 1. \tag{22}$$

Then the difference relations (5) take the form:

$$\begin{aligned} x_{i+1} &= A \cdot \sin\big(v \cdot \Delta t \cdot (i+1)\big), \\ y_{i+1} &= A \cdot \cos\big(v \cdot \Delta t \cdot (i+1)\big), \end{aligned} \quad i = \overline{0, n-1}. \tag{23}$$

The initial values of the sought coordinates of the pursuer $X_i, Y_i, i = \overline{0, n}$ are assumed as follows:

$$X_0 = 0, Y_0 = -1. \tag{24}$$

Then, by performing the calculation using formulas (10), (16), (19) for given values of the number of steps n, step Δt and speeds $v, V (v \neq V)$. three numerical versions of the pursuit curve are obtained.

The implementation of the described algorithm using the Mathcad mathematical package is shown in Figures 8a, b (the animation for Figure 8a is posted on an electronic resource https://community.ptc.com/t5/PTCMathcad/OchkovArticleAnimaions/td-p/583048).

From Figure 8b we notice that the numerical pursuit curves constructed according to the methods (16) and (19) (analogue to the Adams – Bashforth and to the Milne methods respectively) provide a better prediction for the next step of the pursued than the pursuit curve constructed according method (10) (equivalent to the Euler method): the $n+1$ steps of the pursuer trajectories are directed not at the point of the pursued location (as in method (10)), but with anticipation towards the $(n+1)-$ th step of the pursued.

It is obvious that method (19) (equivalent to the Milne method) provides a better approximation to the $(n+1)-$ th step of the pursued than method (16) (equivalent to the Adams – Bashforth method).

Thus, the possibility of using the difference schemes is demonstrated also in the case when the pursued entity moves along an arbitrary trajectory.

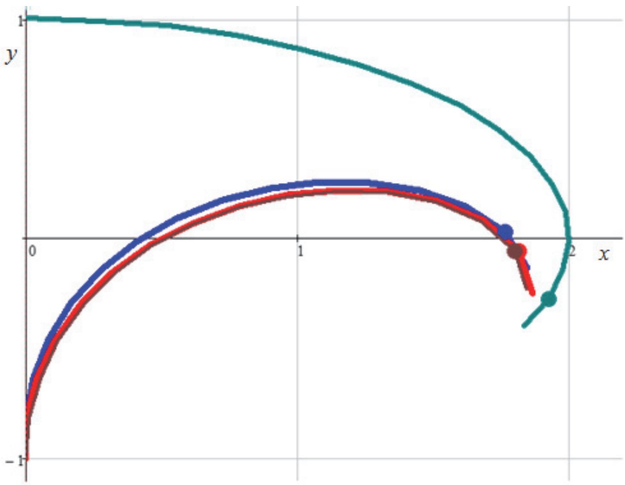


Fig. 8(a). Pursuit curves obtained by different methods for the case of an elliptical path of the pursued: general view

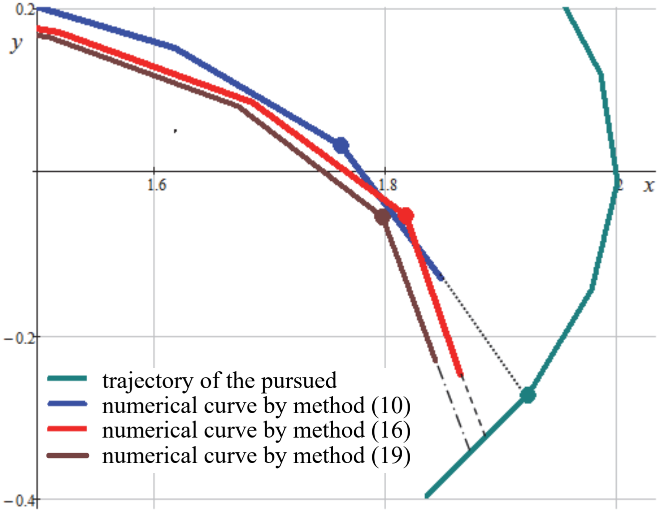


Fig. 8(b). Pursuit curves obtained by different methods for the case of an elliptical path of the pursued: view of part of the curve

The advantage of the proposed method is that it allows to construct numerical interpretations of pursuit curves for given strategies of the pursued, which are not, generally speaking, solutions of any differential equations, as it is usually done in the theory of the differential games (see, for example, [41]).

5. Application of difference schemes for constructing a pursuit curve in the three-dimensional space. The proposed method for constructing difference schemes can be extended to the case of the three-dimensional space. The problem of constructing a pursuit curve in three-dimensional space is most often considered in the context of control theory problems with cyclic pursuit, where several enumerated objects start moving from different points of the space, with each object catching up with the next [42-44]. This case has been studied quite well on the plane (see, for example, [45-47]), and it reduces to the so-called "bugs problem" (this problem is also known as "mice problem"). [48, 49], when the objects begin to move uniformly at the same speed from the vertices of a regular polygon, with each object moving in the direction of its nearest neighbour. In this case, the trajectory of motion of each object is a logarithmic spiral [50].

We construct a difference scheme equivalent to the Euler method for the "bugs problem" in the three-dimensional space, i.e. for the problem of finding the trajectories of objects that begin a uniform motion with equal speed from the vertices of a regular polyhedron.

As a polyhedron, it was considered a tetrahedron with vertices at the points $(0,0,0),\left(\frac{\sqrt{3}}{2},\frac{1}{2},0\right),(0,1,0),\left(\frac{\sqrt{3}}{6},\frac{1}{2},\sqrt{\frac{2}{3}}\right)$. Let four objects begin their uniform motion with the same speed from the vertices of the tetrahedron, with the first object following the second, the second following the third, and so on (Figure 9).

The speeds of the four objects are denoted by $\overline{v_1},...,\overline{v_4}$ (the modules of the speeds are set to be equal). The coordinates of each of the objects in a time-parametric form will be denoted by $\left(x^1(t),y^1(t),z^1(t)\right),\ldots,\left(x^4(t),y^4(t),z^4(t)\right)$.

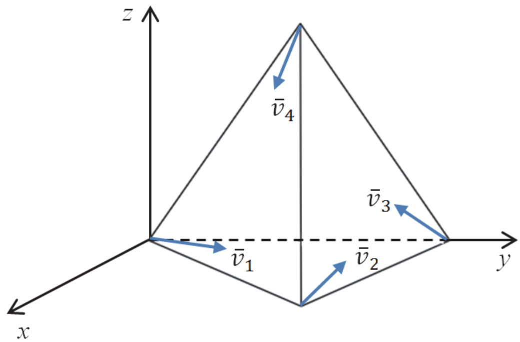


Fig. 9. Cyclic pursuit in a tetrahedron

To construct a difference scheme which is similar to the two-dimensional case, we will not explore the trajectories of the objects of the pursuit in the space itself, but their projections on the $0xy$ and $0xz$ planes.

A partition of the pursuit time into n time intervals is introduced and the time step is denoted by Δt. Then for the coordinates $x^k(t), y^k(t), k=\overline{1,4}$, the difference relation (10) is written for the two-dimensional case in the $0xy$ plane, and for the coordinate $z^k(t), k=\overline{1,4}$ in the $0xz$ plane.

To do this, firstly, it is necessary to calculate, respectively, in the $0xy$ and $0xz$ planes, the angles $\overline{\varphi}_i^{oxy}$, $\overline{\varphi}_i^{oxz}$, $i=\overline{1,n}$, between the tangent to the pursuit curve and the axis Ox by the formula (11).

Secondly, one should use not the modules of objects velocities $\left|\overline{v_k}\right|, k=\overline{1,4}$, but the modules of their projections $\left|\overline{v}_k^{oxy}\right|, \left|\overline{v}_k^{oxz}\right|, k=\overline{1,4}$ on the corresponding planes.

The difference relations are derived for finding the coordinates $\left(x^1(t), y^1(t), z^1(t)\right)$ of the first object, starting from the origin and following the second object with coordinates $\left(x^2(t), y^2(t), z^2(t)\right)$, starting the motion from the vertex $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$. Then the initial values of the coordinates are:

$$\begin{aligned} &x_0^1 = y_0^1 = z_0^1 = 0; \\ &x_0^2 = \frac{\sqrt{3}}{2}, y_0^2 = \frac{1}{2}, z_0^2 = 0. \end{aligned} \qquad (25)$$

According to (10) and taking into account the above remarks, the difference relations are converted to the form:

$$\begin{aligned} &x_{i+1}^1 = x_i^1 + \left|\overline{v}_1^{oxy}\right| \times \Delta t \times \cos(\overline{\varphi}_i^{oxy}), \\ &y_{i+1}^1 = y_i^1 + \left|\overline{v}_1^{oxy}\right| \times \Delta t \times \sin(\overline{\varphi}_i^{oxy}), \\ &z_{i+1}^1 = z_i^1 + \left|\overline{v}_1^{oxz}\right| \times \Delta t \times \sin(\overline{\varphi}_i^{oxz}), \\ &i = \overline{0, n-1}, \end{aligned} \qquad (26)$$

where

$$\overline{\varphi}_i^{oxy} = \begin{cases} \operatorname{arctg}\left(\dfrac{y_i^2 - y_i^1}{x_i^2 - x_i^1}\right) + \pi, \text{for } x_i^1 > x_i^2, \\ \operatorname{arctg}\left(\dfrac{y_i^2 - y_i^1}{x_i^2 - x_i^1}\right), \text{ for } x_i^1 < x_i^2, \\ \dfrac{\pi}{2} \cdot \operatorname{sgn}\left(y_i^2 - y_i^1\right), \text{for } x_i^1 = x_i^2, \end{cases} \qquad (27)$$

$$\overline{\varphi}_i^{oxz} = \begin{cases} \mathrm{arctg}\left(\dfrac{z_i^2 - z_i^1}{x_i^2 - x_i^1}\right) + \pi, \text{for } x_i^1 > x_i^2, \\ \mathrm{arctg}\left(\dfrac{z_i^2 - z_i^1}{x_i^2 - x_i^1}\right), \text{for } x_i^1 < x_i^2, \\ \dfrac{\pi}{2} \cdot \mathrm{sgn}\left(z_i^2 - z_i^1\right), \text{for } x_i^1 = x_i^2, \end{cases}$$

$$i = \overline{0, n}.$$

The projections $\left|\overline{v}_1^{oxy}\right|$, $\left|\overline{v}_1^{oxz}\right|$ are found using the geometric constructions in Figure 10.

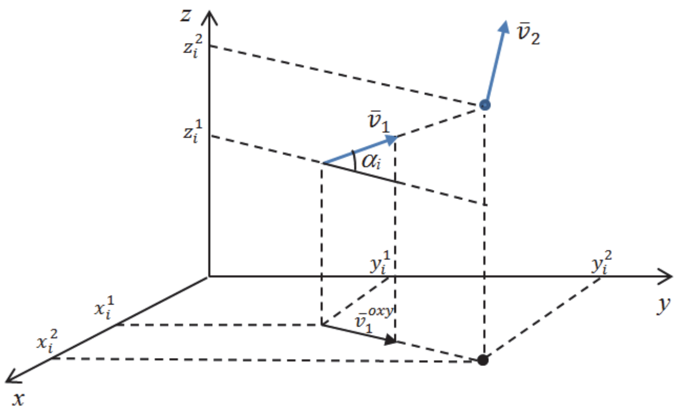


Fig. 10. Determination of the projection of the velocity of an object on the plane

Figure 10 shows that the projection is defined as follows:

$$\left|\overline{v}_1^{oxy}\right| = \left|\overline{v}_1\right| \cdot \cos\left(\alpha_i\right), \tag{28}$$

where a_i is calculated from the relation:

$$tg\left(\alpha_i\right) = \frac{\left|z_i^2 - z_i^1\right|}{\sqrt{\left|x_i^2 - x_i^1\right|^2 + \left|y_i^2 - y_i^1\right|^2}}. \tag{29}$$

Similarly, the projection $\left|\overline{v}_1^{oxz}\right|$ is defined:

$$\left|\overline{v}_1^{oxz}\right| = \left|\overline{v_1}\right| \cdot \cos\left(\beta_i\right), \tag{30}$$

where β_i is calculated from the relationship:

$$\operatorname{tg}\left(\beta_i\right) = \frac{\left|y_i^2 - y_i^1\right|}{\sqrt{\left|x_i^2 - x_i^1\right|^2 + \left|z_i^2 - z_i^1\right|^2}}. \tag{31}$$

Using (28)-(31), the difference method (26) is obtained in the final form:

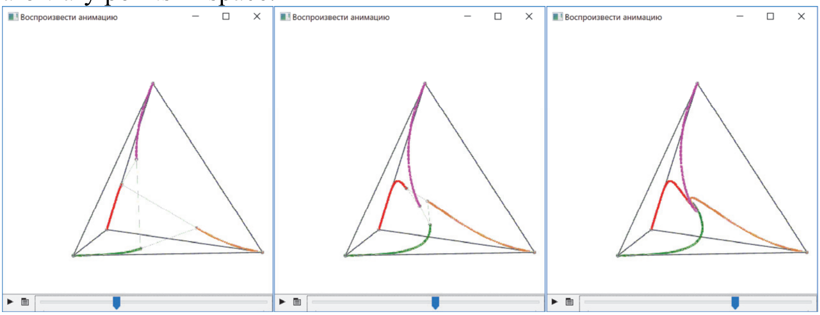
$$\begin{gathered} x_{i+1}^1 = x_i^1 + \left|\overline{v_1}\right| \cdot \cos\left(\alpha_i\right) \cdot \Delta t \cdot \cos(\overline{\varphi}_i^{oxy}), \\ y_{i+1}^1 = y_i^1 + \left|\overline{v_1}\right| \cdot \cos\left(\alpha_i\right) \cdot \Delta t \cdot \sin(\overline{\varphi}_i^{oxy}), \\ z_{i+1}^1 = z_i^1 + \left|\overline{v_1}\right| \cdot \cos\left(\beta_i\right) \cdot \Delta t \cdot \sin(\overline{\varphi}_i^{oxz}), \\ i = \overline{0, n-1}, \end{gathered} \tag{32}$$

where $\overline{\varphi}_i^{oxy}$, $\overline{\varphi}_i^{oxz}$ are determined according to (27), and α_i and β_i are determined by the relations:

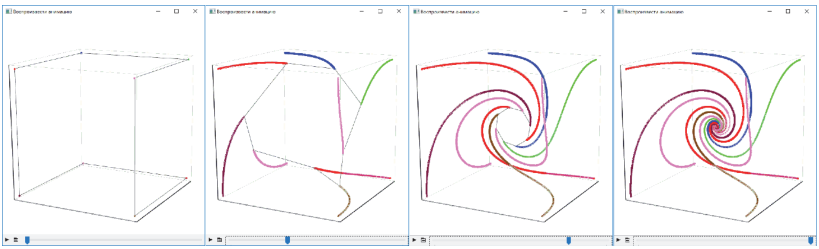
$$\begin{gathered} \alpha_i = \operatorname{arctg}\left(\frac{\left|z_i^2 - z_i^1\right|}{\sqrt{\left|x_i^2 - x_i^1\right|^2 + \left|y_i^2 - y_i^1\right|^2}}\right), \\ \beta_i = \operatorname{arctg}\left(\frac{\left|y_i^2 - y_i^1\right|}{\sqrt{\left|x_i^2 - x_i^1\right|^2 + \left|z_i^2 - z_i^1\right|^2}}\right). \end{gathered} \tag{33}$$

By establishing similar difference relations (32) for the second, third and fourth objects and combining them into a system we obtain a complete difference method for constructing the pursuit curves in the tetrahedron. Figure 11a (animations for Figure 11 are posted on electronic resources https://community.ptc.com/t5/PTC-Mathcad/Is-it-my-own-or-Mathcad-15-error/m-p/576164, https://community.ptc.com/t5/PTC-Mathcad/Batsproblem/m-p/576137) shows the results of the implementation of the difference method in the Mathcad package for the tetrahedron, while Figures 11b, c show the results of implementing similarly constructed pursuit curves for cases when objects start their motion from the vertices of a cube and of a dodecahedron respectively.

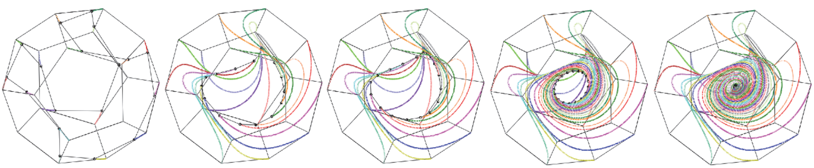
It could be noted that the proposed approach allows to construct difference schemes not only for objects that start their motion with identical speeds from the vertices of Platonic solids, but also for objects that move uniformly with different speeds and begin their motion at arbitrary points in space.



a)



b)



c)

Fig. 11. Constructing the pursuit curves in the three-dimensional space

6. Conclusions. This article proposes a new approach to the construction of pursuit curves through the use of difference schemes. The advantage of the proposed approach is the possibility of describing pursuit curves in a numerical way without deriving the differential equation. The constructed modifications of difference schemes equivalent respectively to the Euler, to the Adams – Bashforth and to the Milne methods approximate the analytical solution with high accuracy. The proposed approach can be applied to the numerical construction of pursuit trajectories with arbitrary pursued strategy either in two-dimensional plans or in three-dimensional spaces.

References

1. Petrosyan L.A., Mazalov V.V. Recent Advances in Game Theory and Applications. Springer International Publishing. 2016. 284 p.
2. Mazalov V., Chirkova J.V. Networking Games. Network Forming Games and Games on Networks. Academic Press. 2019. 322 p.
3. Petrosyan L.A. Differential Games of Pursuit. World Scientific. 1993. 332 p.
4. Isaacs R. Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization. Courier Corporation. 1999. 416 p.
5. Kumkov S.S., Le Menec S., Patsko V.S. Zero-sum pursuit-evasion differential games with many objects: survey of publications. *Dynamic Games and Applications*. 2017. vol. 7. no. 4. pp. 609–633.
6. Ramana M.V., Kothari M. Pursuit-evasion games of high speed evade. *Journal of Intelligent & Robotic Systems*. 2017. vol. 85. no. 2. pp 293–306.
7. Pankov S.Ya., Zaburaev E.Yu., Matveev A.M. *Teoriya i metodika upravleniya aviatsiei* [Theory and Methods of Aviation Control]. Ulyanovsk: UVAU GA. 2006. 190 p. (In Russ.).
8. Barton J.C., Eliezer C.J. On pursuit curves. *The ANZIAM Journal.* 2000. vol. 41. no. 3. pp. 358–371.
9. Samoyavcheva M.V., Fedorov L.I. [The pursuit problem]. *Vestnik moskovskogo gosudarstvennogo oblastnogo universiteta. Seriya: Fizika-matematika ‒ Bulletin of the Moscow Region State University. Series: Physics-Mathematics.* 2011. no. 1. pp. 65–69. (In Russ.).
10. Pták P., Tkadlec J. The Dog-and-Rabbit Chase Revisited. *Acta Polytechnica.* 1996. vol. 36. pp. 5–10.
11. Mungan C.A. A Classic Chase Problem Solved from a Physics Perspective. *European Journal of Physics.* 2005. vol. 26. no. 6. pp. 985–990.
12. Sigaladze Z.K., Chashina O.I. [The problem of pursuing a hare by a wolf as an exercise of elementary kinematics]. *Vestnik NGU. Seriya "Fizika" ‒ Siberian Journal of Physics.* 2010. Issue 5. vol. 2. pp. 111–115. (In Russ.).
13. Pogrebskaya T.N., Soltakhanov Sh.Kh. [The control of chasing a target by the pursuit method as a nonholonomic problem in mechanics]. *Vestnik Sankt-Peterburgskogo universiteta. Seriya 1. Matematika. Mekha-nika. Astronomiya ‒ Vestnik of Saint Petersburg University. Mathematics. Mechanics. Astronomy.* 2007. vol. 1. pp. 117–126. (In Russ.).
14. Kuzmina L.I., Osipov Yu.V. [Calculation of the length of the trajectory for the problem of pursuit]. *Vestnik MGSU ‒ Vestnik MGSU.* 2013. vol 12. pp. 20–26. (In Russ.).
15. Azamov A.A., Kuchkarov A.Sh., Samatov B.O. [The relation between problems of pursuit, controllability and stability in the large in linear systems with different types of constraints]. *Prikladnaya matematika i mehanika – Journal of Applied Mathematics and Mechanics.* 2007. Issue 71. vol. 2. pp. 259–263. (In Russ.).

16. Ivanov A.A., Shmakov O.A. [Algorithm for defining the inner geometry of a snakelike manipulator in case of leading link movements along the incremental trajectory]. *Trudy SPIIRAN – SPIIRAS Proceedings.* 2016. vol. 6(49). pp. 190–207. (In Russ.).
17. Lazarev V.S., Agadjanov D.E. [Using graphic-analytical methods for robots group movement trajectories formation in the two-dimensional environment]. *Trudy SPIIRAN – SPIIRAS Proceedings.* 2016. vol. 2(45). pp. 45–57. (In Russ.).
18. Shan Y. et al. CF-Pursuit: A Pursuit Method with a Clothoid Fitting and a Fuzzy Controller for Autonomous Vehicles. *International Journal of Advanced Robotic Systems.* 2015. vol. 12. no. 9. pp. 134.
19. Amer N.H., Zamzuri H., Hudha K., Kadir Z.A. Modelling and Control Strategies in Path Tracking Control for Autonomous Ground Vehicles: A Review of State of the Art and Challenges. *Journal of Intelligent & Robotic Systems.* 2017. vol. 86. no. 2. pp. 225–254.
20. Spakov O. et al. PursuitAdjuster: an exploration into the design space of smooth pursuit-based widgets. Proceedings of the Ninth Biennial ACM Symposium on Eye Tracking Research & Applications. 2016. pp. 287–290.
21. Khamis M. et al. Eyescout: Active Eye Tracking for Position and Movement Independent Gaze Interaction with Large Public Displays. Proceedings of the 30th Annual ACM Symposium on User Interface Software and Technology (UIST 2017). 2017. pp. 155–166.
22. Bardi M., Falcone M., Soravia P. Numerical Methods for Pursuit-Evasion Games via Viscosity Solutions. Stochastic and Differential Game. 1999. pp. 105–175.
23. Kumkov S., Ménec S., Patsko V. Zero-Sum Pursuit-Evasion Differential Games with Many Objects: Survey of Publications. *Dynamic Games and Applications*. 2017. vol. 7. no. 4. pp. 609–633.
24. Munts N., Kumkov S. [On the coincidence of the minimax solution and the value function in a time-optimal game with a lifeline]. *Trudy Instituta Matematiki i Mekhaniki UrO RAN – Proceedings of Krasovskii Institute of Mathematics and Mechanics*. 2018. vol. 24(2). pp. 200–214. (In Russ.).
25. Alimov K.N., Mamatov M.S. Solving a Pursuit Problem in High-order Controlled Distributed Systems. *Siberian Advances in Mathematics.* 2014. vol. 24(4). pp. 229–239.
26. Kazakov Yu.V. Zadacha o presledovanii na vrashchayushchemsya diske [Chase problem on a turntable]. Available at: http://old.exponenta.ru/EDUCAT/systemat/kasakov/pursuit/index.asp (accessed: 16.03.2019). (In Russ.).
27. Bakhvalov N.S., Zhidkov N.P., Kobelkov G.M. *Chislennyie metodyi* [Numerical Methods]. BINOM. Laboratoriya znaniy. 2008. 636 p. (In Russ.).
28. Ochkov V.F. *Mathcad 14 dlya studentov, inzhenerov u construktorov* [Mathcad 14 for Students, Engineers and Designers]. BHV-Peterburg. 2007. 368 p. (In Russ.).
29. Ochkov V.F., Bogomolova E.P., Ivanov D.A. *Fiziko-matematicheskie etyudy s Mathcad i Internet: Uchebnoe posobie* [Physics and Mathematics Studies with Mathcad and Internet: Tutorial]. Lan Publ. 2018. 560 p. (In Russ.).
30. Ponomarev A.A. [Suboptimal control construction for the model predictive controller]. *Vestnik Sankt-Peterburgskogo universiteta. Seriya 10. Prikladnaya matematika. Informatika. Processy upravleniya – Vestnik of Saint Peterburg University Applied Mathematics. Computer Science. Control Processes.* 2017. vol. 2. pp. 193–208. (In Russ.).
31. Ma Y.J. Reconstruction of a Robin coefficient by a predictor-corrector method. *Mathematical Problems in Engineering.* 2015. vol. 11. pp 1–7.
32. Binder A.J., Luskin M., Ortner C. Analysis of a predictor-corrector method for computationally efficient modeling of surface effects in 1D. 2016. Available at: https://arxiv.org/pdf/1605.05750v1.pdf (accessed: 16.03.2019).

33. Onwubuoya C., Akinyemi S.T., Odabi O.I., Odachi G.N. Numerical simulation of a computer virus transmission model using Euler predictor-corrector method. *IDOSR Journal of Applied Sciences*. 2018. vol. 3(1). pp. 16–28.
34. Abdullahi Y.A., Omar Z., Kuboye J.O. Derivation of block predictor – block corrector method for direct solution of third order ordinary differential equations. *Global Journal of Pure and Applied Mathematics*. 2016. vol. 12(1). pp. 343–350.
35. Daftardar-Gejji V., Sukale Y., Bhalekar S. A new predictor-corrector method for fractional differential equations. *Applied Mathematics and Computation*. 2014. vol. 244. pp. 158–182.
36. Ndanusa A., Tafida F.U. Predictor-corrector methods of high order for numerical integration of initial value problems. *International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*. 2016. vol. 4. no. 2. pp. 47–55.
37. Oghonyon J., Okunuga S.A., Lyase S.A. Milne's implementation on block predictor-corrector methods. *Journal of Applied Sciences*. 2016. vol. 16. no. 5. pp. 236–241.
38. Söderlind G. Multistep Methods. Encyclopedia of Applied and Computational Mathematics. 2015. 10 p.
39. Islam M.A. A comparative study on numerical solutions of initial value problems for ordinary differential equations with Euler and Runge Kutta methods. *American Journal of Computational Mathematics*. 2015. vol. 5. no. 3. pp. 393−404.
40. Fathoni M.F., Wuryandari A.I. Comparison between Euler, Heun, Runge-Kutta and Adams-Bashforth-Moulton integration methods in the particle dynamic simulation. 2015 4th International Conference on Interactive Digital Media (ICIDM). 2015. pp. 1–7.
41. Kuchkarov A.S. Solution of simple pursuit-evasion problem when evader moves on a given curve. *International Game Theory Review*. 2010. vol. 12. no. 3. pp. 223–238.
42. Galloway K.S., Justh E.W., Krishnaprasad P.S. Cyclic pursuit in three dimensions. 49th IEEE Conference on Decision and Control (CDC). 2010. pp. 7141–7146.
43. Galloway K.S., Justh E.W., Krishnaprasad P.S. Geometry of cyclic pursuit. Proceedings of the 48th IEEE Conference on Decision and Control (CDC). 2009. pp. 7485–7490.
44. Mukherjee D., Kumar S. Finite-time heterogeneous cyclic pursuit with application to target interception. 2018. Available at: https://arxiv.org/pdf/1811.10827.pdf (accessed: 16.03.2019).
45. Arnold M., Baryshnikov Y., Liberzon D. Cyclic pursuit without coordinates: convergence to regular polygon formations. 53rd IEEE Conference on Decision and Control. 2014. pp. 6191–6196.
46. Marshall J.A., Brouck M.E., Francis B.A. Pursuit formations of unicycles. *Automatica*. 2006. vol. 42. no. 1. pp. 3–12.
47. Sharma B., Ramakrishnan S., Kumar M. Cyclic pursuit in a multi-agent robotic system with double-integrator dynamics under linear interactions. *Robotica*. 2013. vol. 31. no. 7. pp. 1037–1050.
48. Arnold M., Zharnitsky V. Cyclic evasion in the three bug problem. *The American Mathematical Monthly*. 2015. vol. 122. no. 4. pp. 377–380.
49. Chapman S.J., Lottes J., Trefethen L.N. Four bugs on a rectangle. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. 2010. vol. 467. no. 2127. pp. 881–896.
50. Ding W., Yan G., Lin Z. Formations on Two-Layer Pursuit Systems. 2009 IEEE International Conference on Robotics and Automation. 2009. pp. 3496–3501.

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УДК 519.62 **DOI** 10.15622/sp.2019.18.6.1407-1433

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ПРИМЕНЕНИЕ РАЗНОСТНЫХ СХЕМ К РЕШЕНИЮ ЗАДАЧИ О ПОГОНЕ

Очков В.Ф., Васильева И.Е. **Применение разностных схем к решению задачи о погоне.**

Аннотация. В работе рассматривается один из аспектов задачи о преследовании: построение траекторий движения преследователя для случая, когда преследование осуществляется по методу погони, то есть касательная, проведенная к траектории движения преследователя в любой момент времени, проходит через положение точки, которая ассоциируется с преследуемым. Предлагается новый подход построения кривых погони путем использования разностных схем. Данная методика позволяет отказаться от необходимости составлять дифференциальные уравнения для описания кривых погони, что бывает достаточно сложно сделать в общем случае. Кроме того, применение разностных схем обосновано в ситуации, когда нахождение аналитического решения уже имеющегося дифференциального уравнения затруднительно, и дает возможность получить кривую погони численным способом. Построены различные модификации разностных схем, являющиеся аналогами схем на основе методов Эйлера, Адамса — Башфорта и Милна. Осуществлена их программная реализация с помощью математического пакета Mathcad. Рассмотрен случай равномерного прямолинейного движения преследуемого, для которого известно дифференциальное уравнение, описывающее траекторию преследователя, и его аналитическое решение. Проведен сравнительный анализ полученных разными методами численных решений и известного аналитического решения. Найдена погрешность полученных численных реализаций. Рассмотрено применение построенных разностных схем для более общего случая произвольной траектории преследуемого. Также описан алгоритм распространения предложенного метода для случая циклического преследования с несколькими участниками в трехмерном пространстве. В частности, построена разностная схема, аналогичная методу Эйлера, для трехмерного аналога «задачи о жуках». Полученные результаты продемонстрированы на анимационных примерах как для двумерного, так и трехмерного случаев.

Ключевые слова: дифференциальные игры, задача о преследовании, метод погони, кривая погони, численные методы, разностные схемы, метод Эйлера, «задача о трех жуках», Mathcad.

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Литература

1. *Petrosyan L.A., Mazalov V.V.* Recent Advances in Game Theory and Applications // Springer International Publishing. 2016. 284 p.
2. *Mazalov V., Chirkova J.V.* Networking Games. Network Forming Games and Games on Networks // Academic Press. 2019. 322 p.
3. *Petrosyan L.A.* Differential Games of Pursuit // World Scientific. 1993. 332 p.
4. *Isaacs R.* Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization // Courier Corporation. 1999. 416 p.
5. *Kumkov S.S., Le Menec S., Patsko V.S.* Zero-sum pursuit-evasion differential games with many objects: survey of publications // Dynamic Games and Applications. 2017. vol. 7. no. 4. pp. 609–633.
6. *Ramana M.V., Kothari M.* Pursuit-evasion games of high speed evade // Journal of Intelligent & Robotic Systems. 2017. vol. 85. no. 2. pp. 293–306.
7. *Паньков С.Я., Забураев Е.Ю., Матвеев А.М.* Теория и методика управления авиацией // Ульяновск: УВАУ ГА. 2006. 190 с.
8. *Barton J.C., Eliezer C.J.* On pursuit curves // The ANZIAM Journal. 2000. vol. 41. no. 3. pp. 358–371.
9. *Самоявчева М.В., Федоров Л.И.* Задача о погоне // Вестник Московского государственного университета. Серия: Физика-Математика. 2011. № 1. С. 65–69.
10. *Pták P., Tkadlec J.* The Dog-and-Rabbit Chase Revisited // Acta Polytechnica. 1996. vol. 36. pp. 5–10.
11. *Mungan C.A.* A Classic Chase Problem Solved from a Physics Perspective // European Journal of Physics. 2005. vol. 26. no. 6. pp. 985–990.
12. *Сигаладзе З.К., Чащина О.И.* Задача преследования зайца волком как упражнение элементарной кинематики // Вестник НГУ. Серия: Физика. 2010. Т. 5. № 2. С. 111–115.
13. *Погребская Т.Н., Солтаханов Ш.Х.* Управление преследованием цели по методу погони как неголономная задача механики // Вестник Санкт-Петербургского университета. Серия 1. Математика. Механика. Астрономия. 2007. Вып. 1. С. 117–126.
14. *Кузьмина Л.И., Осипов Ю.В.* Расчет длины траектории для задачи преследования // Вестник МГСУ. 2013. № 12. С. 20–26.
15. *Азамов А.А., Кучкаров А.Ш., Саматов Б.О.* О связи между задачами преследования, управляемости и устойчивости в целом в линейных системах с разнотипными ограничениями // Прикладная математика и механика. 2007. Т. 71. Вып. 2. С. 259–263.
16. *Иванов А.А., Шмаков О.А*. Алгоритм определения внутренней геометрии манипулятора змеевидного типа при движении лидирующего звена по наращиваемой траектории // Труды СПИИРАН. 2016. Вып. 6(49). С. 190–207.
17. *Лазарев В.С., Агаджанов Д.Э.* Использование графоаналитических методов для формирования траектории группы подвижных объектов в двумерной среде // Труды СПИИРАН. 2016. Вып. 2(45). С. 45–57.
18. *Shan Y. et al.* CF-Pursuit: A Pursuit Method with a Clothoid Fitting and a Fuzzy Controller for Autonomous Vehicles // International Journal of Advanced Robotic Systems. 2015. vol. 12. no. 9. pp. 134.
19. *Amer N.H., Zamzuri H., Hudha K., Kadir Z.A.* Modelling and Control Strategies in Path Tracking Control for Autonomous Ground Vehicles: A Review of State of the Art and Challenges // Journal of Intelligent & Robotic Systems. 2017. vol. 86. no. 2. pp. 225–254.
20. *Spakov O. et al.* PursuitAdjuster: an exploration into the design space of smooth pursuit-based widgets // Proceedings of the Ninth Biennial ACM Symposium on Eye Tracking Research & Applications. 2016. pp. 287–290.

21. *Khamis M. et al.* Eyescout: Active Eye Tracking for Position and Movement Independent Gaze Interaction with Large Public Displays // Proceedings of the 30th Annual ACM Symposium on User Interface Software and Technology (UIST 2017). 2017. pp. 155–166.
22. *Bardi M*., *Falcone M*., *Soravia P.* Numerical Methods for Pursuit-Evasion Games via Viscosity Solutions // Stochastic and Differential Game. 1999. pp. 105–175.
23. *Kumkov S., Ménec S., Patsko V.* Zero-Sum Pursuit-Evasion Differential Games with Many Objects: Survey of Publications // Dynamic Games and Applications. 2017. vol. 7. no. 4. pp. 609–633.
24. *Мунц Н.В., Кумков С.С.* О совпадении минимаксного решения и функции цены игры быстродействия с линией жизни // Труды Института математики и механики УрО РАН. 2018. Вып. 24(2). С. 200–214.
25. *Alimov K.N., Mamatov M.S.* Solving a Pursuit Problem in High-order Controlled Distributed Systems // Siberian Advances in Mathematics. 2014. vol. 24(4). pp. 229–239.
26. *Казаков Ю.В.* Задача о преследовании на вращающемся диске. URL: http://old.exponenta.ru/EDUCAT/ systemat/kasakov/pursuit/index.asp (дата обращения: 16.03.2019).
27. *Бахвалов Н.С., Жидков Н.П., Кобельков Г.М.* Численные методы // БИНОМ. Лаборатория знаний. 2008. 636 с.
28. *Очков В.Ф.* Mathcad 14 для студентов, инженеров и конструкторов // БХВ-Петербург. 2007. 368 с.
29. *Очков В.Ф., Богомолова Е.П., Иванов Д.А.* Физико-математические этюды с Mathcad и Интернет: учебное пособие // Лань. 2018. 560 с.
30. *Пономарев А.А.* Аппроксимация обратной связи в регуляторе «Предиктор-корректор» явной функцией // Вестник Санкт-Петербургского университета. Серия 10. Прикладная математика. Информатика. Процессы управления. 2017. № 2. С. 193–208.
31. *Ma Y.J.* Reconstruction of a Robin coefficient by a predictor-corrector method // Mathematical Problems in Engineering. 2015. vol. 11. pp 1–7.
32. *Binder A.J., Luskin M., Ortner C.* Analysis of a predictor-corrector method for computationally efficient modeling of surface effects in 1D. 2016. URL: https://arxiv.org/pdf/1605.05750v1.pdf (дата обращения: 16.03.2019).
33. *Onwubuoya C., Akinyemi S.T., Odabi O.I., Odachi G.N.* Numerical simulation of a computer virus transmission model using Euler predictor-corrector method // IDOSR Journal of Applied Sciences. 2018. vol. 3(1). pp. 16–28.
34. *Abdullahi Y.A., Omar Z., Kuboye J.O.* Derivation of block predictor – block corrector method for direct solution of third order ordinary differential equations // Global Journal of Pure and Applied Mathematics. 2016. vol. 12(1). pp. 343–350.
35. *Daftardar-Gejji V., Sukale Y., Bhalekar S.* A new predictor-corrector method for fractional differential equations // Applied Mathematics and Computation. 2014. vol. 244. pp. 158–182.
36. *Ndanusa A., Tafida F.U.* Predictor-corrector methods of high order for numerical integration of initial value problems // International Journal of Scientific and Innovative Mathematical Research (IJSIMR). 2016. vol. 4. no. 2. pp. 47–55.
37. *Oghonyon J.,Okunuga S.A., Lyase S.A.* Milne's implementation on block predictor-corrector methods // Journal of Applied Sciences. 2016. vol. 16. no. 5. pp. 236–241.
38. *Söderlind G.* Multistep Methods // Encyclopedia of Applied and Computational Mathematics. 2015. 10 p.
39. *Islam M.A.* A comparative study on numerical solutions of initial value problems for ordinary differential equations with Euler and Runge Kutta methods // American Journal of Computational Mathematics. 2015. vol. 5. no. 3. pp. 393–404.

40. *Fathoni M.F., Wuryandari A.I.* Comparison between Euler, Heun, Runge-Kutta and Adams-Bashforth-Moulton integration methods in the particle dynamic simulation // 2015 4th International Conference on Interactive Digital Media (ICIDM). 2015. pp. 1–7.
41. *Kuchkarov A.S.* Solution of simple pursuit-evasion problem when evader moves on a given curve // International Game Theory Review. 2010. vol. 12. no. 3. pp. 223–238.
42. *Galloway K.S., Justh E.W., Krishnaprasad P.S.* Cyclic pursuit in three dimensions // 49th IEEE Conference on Decision and Control (CDC). 2010. pp. 7141–7146.
43. *Galloway K.S., Justh E.W., Krishnaprasad P.S.* Geometry of cyclic pursuit // Proceedings of the 48th IEEE Conference on Decision and Control (CDC). 2009. pp. 7485–7490.
44. *Mukherjee D., Kumar S.* Finite-time heterogeneous cyclic pursuit with application to target interception. 2018. URL: https://arxiv.org/pdf/1811.10827.pdf (дата обращения: 16.03.2019).
45. *Arnold M., Baryshnikov Y., Liberzon D.* Cyclic pursuit without coordinates: convergence to regular polygon formations // 53rd IEEE Conference on Decision and Control. 2014. pp. 6191–6196.
46. *Marshall J.A., Brouck M.E., Francis B.A.* Pursuit formations of unicycles // Automatica. 2006. vol. 42. no. 1. pp. 3–12.
47. *Sharma B., Ramakrishnan S., Kumar M.* Cyclic pursuit in a multi-agent robotic system with double-integrator dynamics under linear interactions // Robotica. 2013. vol. 31. no. 7. pp. 1037–1050.
48. *Arnold M., Zharnitsky V.* Cyclic evasion in the three bug problem // The American Mathematical Monthly. 2015. vol. 122. no. 4. pp. 377–380.
49. *Chapman S.J., Lottes J., Trefethen L.N.* Four bugs on a rectangle // Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 2010. vol. 467. no. 2127. pp. 881–896.
50. *Ding W., Yan G., Lin Z.* Formations on Two-Layer Pursuit Systems // 2009 IEEE International Conference on Robotics and Automation. 2009. pp. 3496–3501.