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Articles

Exact solution method for Fredholm integro-differential equations with multipoint and integral boundary conditions. Part 1. Extention method

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Introduction: *Boundary value problems for differential and integro-differential equations with multipoint and non-local boundary conditions often arise in mechanics, physics, biology, biotechnology, chemical engineering, medical science, finances and other fields. Finding an exact solution of a boundary value problem with Fredholm integro-differential equations is a challenging problem. In most cases, solutions are obtained by numerical methods.* ***Purpose:*** *Search for necessary and sufficient solvability conditions for abstract operator equations and their exact solutions.* ***Results:*** *A direct method is proposed for the exact solution of a certain class of ordinary differential or Fredholm integro-differential equations with separable kernels and multipoint/integral boundary conditions. We study abstract equations of the form* $Bu = Au - \mathbf{g}\mathbf{F}(Au) = f$ *and* $B_1u = A^2u - \mathbf{q}\mathbf{F}(Au) - \mathbf{g}\mathbf{F}(A^2u) = f$ *with non-local boundary conditions* $\Phi(u) = \mathbf{N}\Psi(u)$ *and* $\Phi(u) = \mathbf{N}\Psi(u)$, $\Phi(Au) = \mathbf{D}\mathbf{F}(Au) + \mathbf{N}\Psi(Au)$, *respectively, where A is a differential operator,* $\mathbf{q}$ *and* $\mathbf{g}$ *are vectors,* $\mathbf{D}$ *and* $\mathbf{N}$ *are matrices, and* $\mathbf{F}$, Φ *and* Ψ *are functional vectors. This method is simple to use and can be easily incorporated into any Computer Algebra System (CAS). The upcoming Part 2 of this paper will be devoted to decomposition method for this problem where the operator* B_1 *is quadratic factorable.*

Keywords *— differential and Fredholm integro-differential equations, multipoint and non-local integral boundary conditions, correct operators, exact solutions.*

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Introduction

Boundary value problems (BVP) for differential and integro-differential equations (IDE) with multipoint and nonlocal boundary conditions arise in various fields of mechanics, physics, biology, biotechnology, chemical engineering, medical science, finance and others [1–14]. More precisely these are elasticity, heat and mass transfer, diffraction, underground water flow and population dynamics problems. Perhaps the first known problem which was reduced to the IDE $a_1 y^{iv}(t) + y(t) = -a_2 \int_{-1}^{1} K(t, x) y^{iv}(x) \mathrm{d}x$ is Proctor's problem of Equilibrium of an elastic beam in XVII century. Fredholm integro-differential equations with nonlocal integral boundary conditions and ordinary differential operators, probably, first were considered by J. D. Tamarkin [15]. Problems with nonlocal boundary conditions for elliptic equations first were investigated by A. V. Bitsadze, A. A. Samarskii [16], while BVP for parabolic equations with nonlocal integral boundary conditions were studied by J. R. Cannon [5], L. I. Kamynin [7], N. I. Ionkin [6] and others. Later such investigations for Laplace, Poisson and heat equations were explored by V. A. Il'in and E. L. Moiseev [17] and others [18–20]. Nonlocal BVP involving integral conditions for hyperbolic equations were studied in [21]. Multipoint and nonlocal BVP with integral boundary conditions for ordinary differential equations were considered in [22, 23]. Fractional IDE with integral boundary conditions were given in [24]. The problem of the existence of solutions for nonlocal BVP was the subject of many papers [19, 20, 23, 25–28]. Exact solutions of BVP with Fredholm IDE were considered in [29] and [30]. In most cases numerical methods are employed. Here, the necessary and sufficient solvability conditions of the abstract operator equations:

$$Bu = Au - Qu, \ Qu = \mathbf{g}\mathbf{F}(Au),$$
$$\mathfrak{D}(B) = \{u \in \mathfrak{D}(A) : \Phi(u) = \mathbf{N}\Psi(u)\}; \qquad (1)$$

$$B_1 u = A^2 u - Q_1 u,\ Q_1 u = \mathbf{qF}(Au) + \mathbf{gF}(A^2 u),$$
$$\mathfrak{D}(B_1) = \{u \in \mathfrak{D}(A^2) : \mathbf{\Phi}(u) = \mathbf{N\Psi}(u)\},$$
$$\mathbf{\Phi}(Au) = \mathbf{DF}(Au) + \mathbf{N\Psi}(Au), \tag{2}$$

and their exact solutions are obtained in closed form. This formalism is applied to solve Fredholm IDE with multipoint or nonlocal integral boundary conditions, when A is a differential operator and Q, Q_1 are integral operators with separable kernels. The problems (1), (2) arise naturally from A. A. Dezin, R. O. Oinarov extensions of linear operators [31, 26], which are not restrictions of a maximal operator, unlike the classical M. G. Krein, J. Von. Neuman extensions [32, 33] in Hilbert space and in Banach space [34]. This work is a generalization of the papers [26–28, 35], where integral boundary conditions have not been considered. Solving differential or Fredholm IDE with integral boundary conditions is a complicated problem, since the operators B and B_1 in (1), (2) are obtained by perturbations of boundary conditions and the action of an operator A. Whereas in [26–28, 35] the operators $B = \hat{A} + Q$, $\mathfrak{D}(B) = \mathfrak{D}(\hat{A})$ and $B_1 = \hat{A}^2 + Q_1$, $\mathfrak{D}(B_1) = \mathfrak{D}(\hat{A}^2)$ are obtained only by perturbation of the action of a correct operator $\hat{A}$ which is a restriction of a maximal operator A.

Terminology and notation

Let X, Y be complex Banach spaces and X^* the adjoint space of X, i. e. the set of all complex-valued linear and bounded functionals on X. We denote by $f(x)$ the value of f on x. We write $\mathfrak{D}(A)$ and $R(A)$ for the domain and the range of the operator A, respectively. An operator A_2 is said to be an extension of an operator A_1, or A_1 is said to be a restriction of A_2, in symbol $A_1 \subset A_2$, if $\mathfrak{D}(A_2) \supseteq \mathfrak{D}(A_1)$ and $A_1 x = A_2 x$, for all $x \in \mathfrak{D}(A_1)$. An operator $A: X \to Y$ is called closed if for every sequence x_n in $\mathfrak{D}(A)$ converging to x_0 with $Ax_n \to f_0$, it follows that $x_0 \in \mathfrak{D}(A)$ and $Ax_0 = f_0$. A closed operator A is called maximal if $R(A) = Y$ and $\ker A \neq \{0\}$. An operator $\hat{A} : X \to Y$ is called correct if $R(\hat{A}) = Y$ and the inverse $\hat{A}^{-1}$ exists and is continuous on Y. An operator $\hat{A}$ is called a correct restriction of the maximal operator A if it is a correct operator and $\hat{A} \subset A$. If $\Psi_i \in X^*$, $i = 1, \ldots, n$, then we denote by $\mathbf{\Psi} = col(\Psi_1, \ldots, \Psi_n)$ and $\mathbf{\Psi}(x) = col(\Psi_1(x), \ldots, \Psi_n(x))$. Let $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_n)$ be a vector of X^n. We will denote by $\mathbf{\Psi}(\mathbf{g})$ the $n \times n$ matrix whose i, j-th entry $\mathbf{\Psi}_i(g_j)$ is the value of functional Ψ_i on element g_j. Note that $\mathbf{\Psi}(\mathbf{gC}) = \mathbf{\Psi}(\mathbf{g})\mathbf{C}$, where $\mathbf{C}$ is a $n \times k$ constant matrix. We will also denote by $\mathbf{0}_n$ the zero and by $\mathbf{I}_n$ the identity $n \times n$ matrices. By $\mathbf{0}$ we will denote the zero column vector.

Extension methods for ordinary differential and Fredholm IDE

Let $A : X \xrightarrow{on} X$ be an ordinary m^{th} order differential operator

$$Au(x) = \alpha_0 u^{(m)}(x) + \alpha_1 u^{(m-1)}(x) + \ldots + \alpha_m u(x),$$
$$\alpha_i \in \mathbb{R} \tag{3}$$

and X be a Banach space. Usually $X = \mathbb{C}[a, b]$ or $X = L_p(a, b)$, $p \geq 1$. In the sequel we denote by $X_A^m = \left(D(A), \|\cdot\|_{X^m}\right)$ the Banach space of all m times differentiable functions with norm $\|u(x)\|_{X_A^m} = \sum_{i=0}^{m} \left\|u^{(i)}(x)\right\|_X$ and by X_A^{m-1} the Banach space of all $m-1$ times differentiable functions with norm

$$\|u(x)\|_{X_A^{m-1}} = \sum_{i=0}^{m-1} \left\|u^{(i)}(x)\right\|_X. \tag{4}$$

Note that for $X = \mathbb{C}[a, b]$ the spaces X_A^m, X_A^{m-1} are defined by $\mathbb{C}^m[a, b]$, $\mathbb{C}^{m-1}[a, b]$, respectively. It is a well-known fact that the operator defined by

$$\hat{A}u(x) = \alpha_0 u^{(m)}(x) +$$
$$+ \alpha_1 u^{(m-1)}(x) + \ldots + \alpha_m u(x) = f,$$
$$\alpha_i \in \mathbb{R},\ x \in [a, b], \tag{5}$$
$$\mathfrak{D}(\hat{A}) =$$
$$= \left\{u(x) \in \mathbb{C}^m[a, b] : u(a) = u'(a) = \ldots = u^{m-1}(a) = 0\right\}$$

is a correct restriction of A and the unique solution of (5) is

$$u(x) = \hat{A}^{-1} f(x) = \frac{1}{(m-1)!} \int_a^x (x-t)^{m-1} f(t)\mathrm{d}t,$$
$$f(x) \in \mathbb{C}[a, b]. \tag{6}$$

Lemma 1. Let $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}$ are $n \times n$ matrices, where $i = 1, 2, 3$, and $\mathbf{G} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 \end{pmatrix}$. Then the next properties of determinants hold true:

$$\det \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 \end{pmatrix} =$$
$$= \det \begin{pmatrix} \mathbf{A}_1 \pm \mathbf{DB}_1 & \mathbf{A}_2 \pm \mathbf{DB}_2 & \mathbf{A}_3 \pm \mathbf{DB}_3 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 \end{pmatrix}; \tag{7}$$

$$\det\begin{pmatrix}\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3\\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3\\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3\end{pmatrix}=\det\begin{pmatrix}\mathbf{A}_1 & \mathbf{A}_2\pm\mathbf{A}_3\mathbf{D} & \mathbf{A}_3\\ \mathbf{B}_1 & \mathbf{B}_2\pm\mathbf{B}_3\mathbf{D} & \mathbf{B}_3\\ \mathbf{C}_1 & \mathbf{C}_2\pm\mathbf{C}_3\mathbf{D} & \mathbf{C}_3\end{pmatrix}. \quad (8)$$

Proof: Let $\mathbf{H}=\begin{pmatrix}\mathbf{I}_n & -\mathbf{D} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{I}_n\end{pmatrix}$. Then $\mathbf{H}^{-1}=$ $=\begin{pmatrix}\mathbf{I}_n & \mathbf{D} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{I}_n\end{pmatrix}$, $\det\mathbf{H}=\det\mathbf{H}^{-1}=1$, $|\mathbf{HG}|=|\mathbf{H}||\mathbf{G}|=$ $=|\mathbf{G}|$ and $|\mathbf{H}^{-1}\mathbf{G}|=|\mathbf{H}^{-1}||\mathbf{G}|=|\mathbf{G}|$. So (7) holds.

Let now $\mathbf{H}=\begin{pmatrix}\mathbf{I}_n & \mathbf{0}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{D} & \mathbf{I}_n\end{pmatrix}$. Then $\mathbf{H}^{-1}=$ $=\begin{pmatrix}\mathbf{I}_n & \mathbf{0}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{0}_n\\ \mathbf{0}_n & -\mathbf{D} & \mathbf{I}_n\end{pmatrix}$, $|\mathbf{H}|=|\mathbf{H}^{-1}|=1$, $|\mathbf{GH}|=|\mathbf{G}||\mathbf{H}|=|\mathbf{G}|$ and $|\mathbf{GH}^{-1}|=|\mathbf{G}||\mathbf{H}^{-1}|=|\mathbf{G}|$. So (8) holds and Lemma 1 is proved.

Remark 1. Consider a $n^2\times n^2$ matrix $\mathbf{G}=$ $=\begin{pmatrix}\mathbf{A}_{11} & \dots & \mathbf{A}_{1n}\\ \dots & \dots & \dots\\ \mathbf{A}_{n1} & \dots & \mathbf{A}_{nn}\end{pmatrix}$, where $\mathbf{A}_{ij}$, $i, j = 1, \dots, n$ are $n\times n$ matrices. Let $\mathbf{\Gamma}$ be the matrix obtained from $\mathbf{G}$ by multiplying from the left a row by the $n\times n$ matrix $\mathbf{D}$ and then adding it to another row, or by multiplying from the right a column of $\mathbf{G}$ by the matrix $\mathbf{D}$ and then adding it to another column of $\mathbf{G}$. Then $\det\mathbf{G}=\det\mathbf{\Gamma}$.

Theorem 1. Let X be a complex Banach space, $A: X\xrightarrow{on}X$ an operator from (3) with finite dimensional kernel $\mathbf{z}=(z_1, \dots, z_m)$ which is a basis of $\ker A$, and let $\hat{A}$ be a correct restriction of A defined by

$$\hat{A}\subset A,$$
$$\mathfrak{D}(\hat{A})=\{u\in\mathfrak{D}(A):\mathbf{\Phi}(u)=\mathbf{0}\}, \quad (9)$$

the components of the functional vectors $\mathbf{\Phi}=col\ (\Phi_1, \dots, \Phi_m)$, $\mathbf{\Psi}=col\ (\Psi_1, \dots, \Psi_n)$ and $\mathbf{F}=$ $=col\ (F_1, \dots, F_n)$ belong to X^{m-1} and respectively.

Suppose also that $\Phi_1, \dots, \Phi_m$ biorthogonal to $z_1, \dots, z_m$ and that the components of vector $\mathbf{g}=(g_1, \dots, g_n)\in X^n$ are linearly independent and $\mathbf{N}$ is a $m\times n$ matrix. Then:

(i) The operator B defined by

$$Bu=Au-\mathbf{gF}(Au)=f,$$
$$f\in X;$$
$$\mathfrak{D}(B)=\{u\in\mathfrak{D}(A):\mathbf{\Phi}(u)=\mathbf{N\Psi}(u)\} \quad (10)$$

is injective if and only if

$$\det\mathbf{V}=\det[\mathbf{I}_n-\mathbf{\Psi}(\mathbf{z})\mathbf{N}]\neq 0 \text{ and}$$
$$\det\mathbf{W}=\det[\mathbf{I}_n-\mathbf{F}(\mathbf{g})]\neq 0. \quad (11)$$

(ii) If B is injective, then B is correct and for all $f\in X$ the unique solution of (10) is given by

$$u=B^{-1}f=\hat{A}^{-1}f+\left[\hat{A}^{-1}\mathbf{g}+\mathbf{zNV}^{-1}\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\right]\times$$
$$\times\mathbf{W}^{-1}\mathbf{F}(f)+\mathbf{zNV}^{-1}\mathbf{\Psi}\left(\hat{A}^{-1}f\right). \quad (12)$$

Proof: (i). Let $\det\mathbf{W}\neq 0$, $\det\mathbf{V}\neq 0$ and $u\in\ker B$. Then $Bu=Au-\mathbf{gF}(Au)=0$, $\mathbf{\Phi}(u)=\mathbf{N\Psi}(u)$ and $[\mathbf{I}_n-\mathbf{F}(\mathbf{g})]\mathbf{F}(Au)=\mathbf{0}$, $\mathbf{\Phi}(u-\mathbf{zN\Psi}(u))=0$. The last equation, since (9), implies $u-\mathbf{zN\Psi}(u)\in\mathfrak{D}(\hat{A})$. From $[\mathbf{I}_n-\mathbf{F}(\mathbf{g})]\mathbf{F}(Au)=\mathbf{0}$, since $\det W\neq 0$, follows $\mathbf{F}(Au)=\mathbf{0}$. Then $Bu=Au=0$ which yelds $\hat{A}(u-\mathbf{zN\Psi}(u))=0$ and so $u=\mathbf{zN\Psi}(u)$. Then $\mathbf{\Psi}(u)=\mathbf{\Psi}(\mathbf{z})\mathbf{N\Psi}(u)$ or $[\mathbf{I}_n-\mathbf{\Psi}(\mathbf{z})\mathbf{N}]\mathbf{\Psi}(u)=\mathbf{0}$. The last, since $\det\mathbf{V}\neq 0$ implies $\mathbf{\Psi}(u)=\mathbf{0}$ and so from $u=\mathbf{zN\Psi}(u)$ we get $u=0$, i. e. $\ker B=\{0\}$ and B is an injective operator.

Conversely. Let $\det\mathbf{V}=0$. Then there exists a vector $\mathbf{c}=col\ (c_1, \dots, c_n)=0$ such that $\mathbf{Vc}=0$.

Consider the element $u_0=\mathbf{zNc}\neq 0$, otherwise $\mathbf{Nc}=\mathbf{0}$ and from $[\mathbf{I}_n-\mathbf{\Psi}(\mathbf{z})\mathbf{N}]\mathbf{c}=\mathbf{0}$ follows $\mathbf{c}=\mathbf{0}$, which contradicts the hypothesis $\mathbf{c}\neq\mathbf{0}$. Note that $u_0\in\mathfrak{D}(B)$, since $\mathbf{\Phi}(u_0)=\mathbf{Nc}$, $\mathbf{\Psi}(u_0)=\mathbf{\Psi}(\mathbf{z})\mathbf{Nc}$, $\mathbf{\Phi}(u_0)-\mathbf{N\Psi}(u_0)=\mathbf{Nc}-\mathbf{N\Psi}(\mathbf{z})\mathbf{Nc}=\mathbf{N}[\mathbf{I}_n-\mathbf{\Psi}(\mathbf{z})\mathbf{N}]\mathbf{c}=$ $=\mathbf{NVc}=\mathbf{0}$. It is evident that $u_0\in\ker B$. So $u_0\in\ker B$. Hence $\ker B\neq\{0\}$ and B is not injective. Let now $\det\mathbf{V}\neq 0$, but $\det\mathbf{W}=0$. Then there exists a vector $\mathbf{c}=col(c_1, \dots, c_n)\neq\mathbf{0}$ such that $\mathbf{Wc}=0$. Note that $\mathbf{gc}\neq 0$ because of $g_1, \dots, g_n$ is a linearly independent set and that the element $u_0=\left[\hat{A}^{-1}\mathbf{g}+\mathbf{zNV}^{-1}\mathbf{\Psi}\left(A^{-1}\mathbf{g}\right)\right]\mathbf{c}\neq 0$, otherwise $\mathbf{g}=\mathbf{0}$. For u_0 we obtain

$$u_0=\left[\hat{A}^{-1}\mathbf{g}+\mathbf{zNV}^{-1}\mathbf{\Psi}\left(A^{-1}\mathbf{g}\right)\right]\mathbf{c}\neq 0,$$
$$\mathbf{\Phi}(u_0)-\mathbf{N\Psi}(u_0)=\mathbf{NV}^{-1}\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}-\mathbf{N\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}-$$
$$-\mathbf{N\Psi}(\mathbf{z})\mathbf{NV}^{-1}\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}=$$
$$=\mathbf{N}\left[\mathbf{I}_n-\mathbf{\Psi}(\mathbf{z})\mathbf{N}\right]\mathbf{V}^{-1}\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}-\mathbf{N\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}=$$
$$=\mathbf{N\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}-\mathbf{N\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}=\mathbf{0},$$
$$Bu_0=Au_0-\mathbf{gF}\left(\hat{A}u_0\right)=$$
$$=\mathbf{gc}-\mathbf{gF}(\mathbf{g})\mathbf{c}=\mathbf{g}\left[\mathbf{I}_n-\mathbf{F}(\mathbf{g})\right]\mathbf{c}=\mathbf{gWc}=\mathbf{g0}=0.$$

So $u_0\in\ker B$. Consequently $\ker B\neq\{0\}$ and B is not injective. Hence B is injective if and only if $\det\mathbf{V}\neq 0, \det\mathbf{W}\neq 0$. The statement (i) holds.

(ii) Let $\det\mathbf{W} \neq 0$ and $\det\mathbf{V} \neq 0$. By statement (i), the operator B is injective. Since $\mathbf{z} \in [\ker A]^m$, $\mathbf{\Phi}(\mathbf{z}) = \mathbf{I}_m$, the problem (10) is written as

$$Bu = A(u - \mathbf{zN\Psi}(u)) - \mathbf{gF}(Au) = f,$$

$$f \in X;$$

$$\mathfrak{D}(B) = \{u \in \mathfrak{D}(A) : \mathbf{\Phi}(u - \mathbf{zN\Psi}(u)) = 0\}. \quad (13)$$

Then, applying Equation (9) and relation $Bu = \hat{A}(u - \mathbf{zN\Psi}(u)) - \mathbf{gF}(Au) = f$ we obtain $u - \mathbf{zN\Psi}(u) \in \mathfrak{D}(\hat{A})$, $Bu = \hat{A}(u - \mathbf{zN\Psi}(u)) - \mathbf{gF}(Au) = f$ and for every $u \in \mathfrak{D}(B)$, $f \in X$ using (10), (13) we obtain

$$[\mathbf{I}_n - \mathbf{F}(\mathbf{g})]\mathbf{F}(Au) = \mathbf{F}(f),$$

$$\mathbf{F}(Au) = \mathbf{W}^{-1}\mathbf{F}(f),$$

$$u - \mathbf{zN\Psi}(u) = \hat{A}^{-1}\mathbf{gF}(Au) + \hat{A}^{-1}f,$$

$$\mathbf{\Psi}(u) - \mathbf{\Psi}(\mathbf{z})\mathbf{N\Psi}(u) = \mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{F}(Au) + \mathbf{\Psi}(\hat{A}^{-1}f),$$

$$[\mathbf{I}_n - \mathbf{\Psi}(\mathbf{z})\mathbf{N}]\mathbf{\Psi}(u) = \mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{W}^{-1}\mathbf{F}(f) + \mathbf{\Psi}(\hat{A}^{-1}f),$$

$$\mathbf{\Psi}(u) = \mathbf{V}^{-1}\left[\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{W}^{-1}\mathbf{F}(f) + \mathbf{\Psi}(\hat{A}^{-1}f)\right],$$

$$u = B^{-1}f = \hat{A}^{-1}f + \hat{A}^{-1}\mathbf{gW}^{-1}\mathbf{F}(f) +$$

$$+\mathbf{zNV}^{-1}\left[\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{W}^{-1}\mathbf{F}(f) + \mathbf{\Psi}(\hat{A}^{-1}f)\right].$$

From the last equation for every $f \in X$ follows the unique solution (12) of (10). Because f in (12) is arbitrary, we obtain $R(B) = X$. Since the operator $\hat{A}^{-1}$ and the functionals $F_1, \ldots, F_n, \Psi_1, \ldots, \Psi_n$ are bounded, from (12) follows the boundedness of B^{-1}. Hence, the operator B is correct if and only if (11) holds and the unique solution of (10) is given by (12). The theorem is proved.

From the previous theorem for $\mathbf{g} = \mathbf{0}$ follows the next corollary which is useful for solving some classes of differential equations with nonlocal boundary conditions.

Corollary 1. Let a complex Banach space X, the operators A, $\hat{A}$, the vector $\mathbf{z}$ and functional vectors $\mathbf{\Phi}, \mathbf{\Psi}$ and the matrix $\mathbf{N}$ be defined as in Theorem 1. Then:

(i) The operator B defined by

$$Bu = Au = f,$$

$$f \in X;$$

$$\mathfrak{D}(B) = \{u \in \mathfrak{D}(A) : \mathbf{\Phi}(u) = \mathbf{N\Psi}(u)\} \quad (14)$$

is correct if and only if $\det\mathbf{V} = \det[\mathbf{I}_n - \mathbf{\Psi}(\mathbf{z})\mathbf{N}] \neq 0$ and for all $f \in X$ the unique solution of (14) is given by

$$u = B^{-1}f = \hat{A}^{-1}f + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f). \quad (15)$$

Theorem 2. Let a Banach space X, the vectors $\mathbf{z}$, $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{F}$, the operators A, $\hat{A}$ be defined as in Theorem 1 and the operator $B_1: X \to X$ by

$$B_1u = A^2u - \mathbf{qF}(Au) - \mathbf{gF}(A^2u) = f; \quad (16)$$

$$\mathfrak{D}(B_1) = \{u \in \mathfrak{D}(A^2) : \mathbf{\Phi}(u) = \mathbf{N\Psi}(u),$$

$$\mathbf{\Phi}(Au) = \mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\}. \quad (17)$$

Suppose also that the vectors $\mathbf{q}$ and $\mathbf{g}$ are linearly independent, $\mathbf{q} = (q_1, \ldots, q_n)$, $\mathbf{g} = (g_1, \ldots, g_n) \in X^n$, and $\mathbf{D}$, $\mathbf{N}$ are $m \times n$ matrices. Then:

(i) The operator B_1 corresponding to the problem (16), (17) is injective if and only if

$$\det\mathbf{L} =$$

$$= \det\begin{pmatrix} \mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{K}_1 & -\mathbf{F}(\hat{A}^{-1}\mathbf{g}) \\ \mathbf{V} & -\mathbf{\Psi}(\hat{A}^{-1}\mathbf{z})\mathbf{N} & -\mathbf{K}_3 & -\mathbf{\Psi}(\hat{A}^{-2}\mathbf{g}) \\ \mathbf{0}_n & \mathbf{V} & -\mathbf{K}_2 & -\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g}) \\ \mathbf{0}_n & \mathbf{0}_n & -\mathbf{F}(\mathbf{q}) & \mathbf{W} \end{pmatrix} \neq 0, \quad (18)$$

where

$$\mathbf{K}_1 = \mathbf{I}_n - \mathbf{F}(\mathbf{z})\mathbf{D} - \mathbf{F}(\hat{A}^{-1}\mathbf{q}),\ \mathbf{K}_2 = \mathbf{\Psi}(\mathbf{z})\mathbf{D} + \mathbf{\Psi}(\hat{A}^{-1}\mathbf{q}),$$

$$\mathbf{K}_3 = \mathbf{\Psi}(\hat{A}^{-1}\mathbf{z})\mathbf{D} + \mathbf{\Psi}(\hat{A}^{-2}\mathbf{q}),$$

$$\mathbf{W} = \mathbf{I}_n - \mathbf{F}(\mathbf{g}),\ \mathbf{V} = \mathbf{I}_n - \mathbf{\Psi}(\mathbf{z})\mathbf{N}. \quad (19)$$

(ii) If the operator B_1 is injective, then it is correct and the unique solution of (16), (17) is given by

$$u = B_1^{-1}f = \hat{A}^{-2}f +$$

$$+\left(\mathbf{zN},\ \hat{A}^{-1}\mathbf{zN},\ \hat{A}^{-1}\mathbf{zD} + \hat{A}^{-2}\mathbf{q},\ \hat{A}^{-2}\mathbf{g}\right)\times$$

$$\times\mathbf{L}^{-1}col\left(\mathbf{F}(\hat{A}^{-1}f),\ \mathbf{\Psi}(\hat{A}^{-2}f),\ \mathbf{\Psi}(\hat{A}^{-1}f),\ \mathbf{F}(f)\right). \quad (20)$$

Proof: (i) Let $\det\mathbf{L} \neq 0$. Since $\mathbf{\Phi}(\mathbf{z}) = \mathbf{I}_m$, the relations (17) can be represented as

$$\mathbf{\Phi}(u - \mathbf{zN\Psi}(u)) = 0,$$

$$\mathbf{\Phi}(Au - \mathbf{zDF}(Au) - \mathbf{zN\Psi}(Au)) = 0,$$

which taking into account (9) imply

$$u - \mathbf{zN\Psi}(u) \in \mathfrak{D}(\hat{A}); \quad (21)$$

$$Au - \mathbf{zDF}(Au) - \mathbf{zN\Psi}(Au) \in \mathfrak{D}(\hat{A}). \quad (22)$$

Then, since $\mathbf{z} \in [\ker A]^m$, $\hat{A} \subset A$ and $\hat{A}$ is correct, from (16) we obtain

$$\hat{A}\left(Au - \mathbf{z}[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)]\right) -$$

$$-\mathbf{qF}(Au) - \mathbf{gF}(A^2u) = f,$$

$$Au - \mathbf{z}\left[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\right] - \hat{A}^{-1}\mathbf{qF}(Au) -$$
$$-\hat{A}^{-1}\mathbf{gF}\left(A^2u\right) = \hat{A}^{-1}f,$$

$$\hat{A}\left(u - \mathbf{zN\Psi}(u)\right) - \mathbf{z}\left[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\right] -$$
$$-\hat{A}^{-1}\mathbf{qF}(Au) - \hat{A}^{-1}\mathbf{gF}\left(A^2u\right) = \hat{A}^{-1}f,$$

$$u - \mathbf{zN\Psi}(u) - \hat{A}^{-1}\mathbf{z}\left[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\right] -$$
$$-\hat{A}^{-2}\mathbf{qF}(Au) - \hat{A}^{-2}\mathbf{gF}\left(A^2u\right) = \hat{A}^{-2}f.$$

Then taking into account (16) we get

$$A^2u = \mathbf{qF}(Au) + \mathbf{gF}(A^2u) + f,$$

$$Au = \mathbf{z}\left[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\right] +$$
$$+\hat{A}^{-1}\mathbf{qF}(Au) + \hat{A}^{-1}\mathbf{gF}\left(A^2u\right) + \hat{A}^{-1}f,$$

$$u = \mathbf{zN\Psi}(u) + \hat{A}^{-1}\mathbf{zN\Psi}(Au) +$$
$$+\left(\hat{A}^{-1}\mathbf{zD} + \hat{A}^{-2}\mathbf{q}\right)\mathbf{F}(Au) + \hat{A}^{-2}\mathbf{gF}\left(A^2u\right) + \hat{A}^{-2}f. \quad (23)$$

Further acting by functionals $\mathbf{F}$ and $\mathbf{\Psi}$ we get the next system

$$\mathbf{F}(Au) = \mathbf{F}(\mathbf{z})\left[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\right] +$$
$$+\mathbf{F}\left(\hat{A}^{-1}\mathbf{q}\right)\mathbf{F}(Au) + \mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{F}\left(A^2u\right) + \mathbf{F}\left(\hat{A}^{-1}f\right),$$

$$\mathbf{\Psi}(u) = \mathbf{\Psi}(\mathbf{z})\mathbf{N\Psi}(u) + \mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N\Psi}(Au) +$$
$$+\left[\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{D} + \mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{q}\right)\right]\mathbf{F}(Au) +$$
$$+\mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\mathbf{F}\left(A^2u\right) + \mathbf{\Psi}\left(\hat{A}^{-2}f\right),$$

$$\mathbf{\Psi}(Au) = \mathbf{\Psi}(\mathbf{z})\left[\mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\right] +$$
$$+\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{q}\right)\mathbf{F}(Au) + \mathbf{\Psi}\left(A^{-1}\mathbf{g}\right)\mathbf{F}\left(A^2u\right) + \mathbf{\Psi}\left(\hat{A}^{-1}f\right),$$

$$\mathbf{F}(A^2u) = \mathbf{F}(\mathbf{q})\mathbf{F}(Au) + \mathbf{F}(\mathbf{g})\mathbf{F}(A^2u) + \mathbf{F}(f), \text{ or}$$

$$-\mathbf{F}(\mathbf{z})\mathbf{N\Psi}(Au) + \left[\mathbf{I}_n - \mathbf{F}(\mathbf{z})\mathbf{D} - \mathbf{F}\left(\hat{A}^{-1}\mathbf{q}\right)\right]\mathbf{F}(Au) -$$
$$-\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{F}\left(A^2u\right) = \mathbf{F}\left(\hat{A}^{-1}f\right),$$

$$\mathbf{V\Psi}(u) - \mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N\Psi}(Au) -$$
$$-\left[\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{D} + \mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{q}\right)\right]\mathbf{F}(Au) -$$
$$-\mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\mathbf{F}\left(A^2u\right) = \mathbf{\Psi}\left(\hat{A}^{-2}f\right),$$

$$\mathbf{V\Psi}(Au) - \left[\mathbf{\Psi}(\mathbf{z})\mathbf{D} + \mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{q}\right)\right]\mathbf{F}(Au) -$$
$$-\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{F}\left(A^2u\right) = \mathbf{\Psi}\left(\hat{A}^{-1}f\right),$$

$$-\mathbf{F}(\mathbf{q})\mathbf{F}(Au) + [\mathbf{I}_n - \mathbf{F}(\mathbf{g})]\mathbf{F}(A^2u) = \mathbf{F}(f).$$

Using the notations (19) from the above equations we get the system

$$\begin{pmatrix} \mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{K}_1 & -\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right) \\ \mathbf{V} & -\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N} & -\mathbf{K}_3 & -\mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right) \\ \mathbf{0}_n & \mathbf{V} & -\mathbf{K}_2 & -\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right) \\ \mathbf{0}_n & \mathbf{0}_n & -\mathbf{F}(\mathbf{q}) & W \end{pmatrix} \times$$
$$\times \begin{pmatrix} \mathbf{\Psi}(u) \\ \mathbf{\Psi}(Au) \\ \mathbf{F}(Au) \\ \mathbf{F}\left(A^2u\right) \end{pmatrix} = \begin{pmatrix} \mathbf{F}\left(\hat{A}^{-1}f\right) \\ \mathbf{\Psi}\left(\hat{A}^{-2}f\right) \\ \mathbf{\Psi}\left(\hat{A}^{-1}f\right) \\ \mathbf{F}(f) \end{pmatrix}. \quad (24)$$

Let $u \in \ker B_1$. Then in the systems (23), (24) $f = 0$ and from (24) we get $\mathbf{L}col(\mathbf{\Psi}(u), \mathbf{\Psi}(Au), \mathbf{F}(Au), \mathbf{F}(A^2u)) = \mathbf{0}$, which since $\det\mathbf{L} \neq 0$, yields $\mathbf{\Psi}(u) = \mathbf{\Psi}(Au) = \mathbf{F}(Au) = \mathbf{F}(A^2u) = \mathbf{0}$. Substitution of these values into (16), (17) imply $B_1u = A^2u = 0$, $\mathbf{\Phi}(u) = \mathbf{\Phi}(Au) = \mathbf{0}$. Taking into account (9) we acquire $u \in \mathfrak{D}\left(\hat{A}^2\right)$ and $B_1u = \hat{A}^2u = 0$. By hypothesis $\hat{A}$ is correct and so $u = 0$. Thus $\ker B_1 = \{0\}$ and B_1 is injective.

Conversely. Let $\det\mathbf{L} = 0$. Then there exists a vector $\mathbf{c} = col(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4)$, where $\mathbf{c}_i = col(c_{i1}, \ldots, c_{in})$, $i = 1, \ldots, 4$ such that $\mathbf{c} \neq \mathbf{0}$ and $\mathbf{Lc} = \mathbf{0}$, which since (24) yields

$$-\mathbf{F}(\mathbf{z})\mathbf{Nc}_2 + \mathbf{K}_1\mathbf{c}_3 - \mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4 = \mathbf{0}; \quad (25)$$

$$\mathbf{Vc}_1 - \mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{Nc}_2 - \mathbf{K}_3\mathbf{c}_3 - \mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\mathbf{c}_4 = \mathbf{0}; \quad (26)$$

$$\mathbf{Vc}_2 - \mathbf{K}_2\mathbf{c}_3 - \mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4 = \mathbf{0}; \quad (27)$$

$$-\mathbf{F}(\mathbf{q})\mathbf{c}_3 + \mathbf{Wc}_4 = \mathbf{0}. \quad (28)$$

Consider the element

$$u_0 = \mathbf{zNc}_1 + \hat{A}^{-1}\mathbf{z}(\mathbf{Nc}_2 + \mathbf{Dc}_3) + \hat{A}^{-2}(\mathbf{qc}_3 + \mathbf{gc}_4). \quad (29)$$

Note that $u_0 \neq \mathbf{0}$, otherwise because of the linear independence of the vectors $\mathbf{q}$, $\mathbf{g}$, $\mathbf{z}$ and $\mathbf{D}\left(\hat{A}\right) \cap \ker A = \{0\}$ [18], we get $\mathbf{Nc}_1 = \mathbf{Nc}_2 = \mathbf{c}_3 = \mathbf{c}_4 = \mathbf{0}$. Then from (27) follows that $\mathbf{c}_2 = \mathbf{0}$ and from (26) we obtain $\mathbf{c}_1 = \mathbf{0}$. Thus $\mathbf{c}_i = \mathbf{0}$, $i = 1, \ldots, 4$ and $\mathbf{c} = \mathbf{0}$. But the last contradicts the hypothesis $\mathbf{c} \neq \mathbf{0}$. So $u_0 \neq 0$. From (29), since $\mathbf{\Phi}(\mathbf{z}) = \mathbf{I}_m$, $\mathbf{K}_3 = \mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{D} + \mathbf{\Psi}\left(\hat{A}^{-2}\mathbf{q}\right)$ and (26) we get

$$Au_0 = \mathbf{z}(\mathbf{Nc}_2 + \mathbf{Dc}_3) + \hat{A}^{-1}(\mathbf{qc}_3 + \mathbf{gc}_4),$$

$$A^2u_0 = \mathbf{qc}_3 + \mathbf{gc}_4,$$

$$\mathbf{\Phi}(u_0) - \mathbf{N\Psi}(u_0) = \mathbf{Nc}_1 - \mathbf{N\Psi}(\mathbf{z})\mathbf{Nc}_1 - \mathbf{N\Psi}\left(\hat{A}^{-1}\mathbf{z}\right) \times$$
$$\times(\mathbf{Nc}_2 + \mathbf{Dc}_3) - \mathbf{N\Psi}\left(\hat{A}^{-2}\mathbf{q}\right)\mathbf{c}_3 - \mathbf{N\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\mathbf{c}_4 =$$

$$=\mathbf{N}\left[\mathbf{V}\mathbf{c}_1-\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N}\mathbf{c}_2-\mathbf{K}_3\mathbf{c}_3-\boldsymbol{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\mathbf{c}_4\right]=$$
$$=\mathbf{N}\mathbf{0}=\mathbf{0}.$$

Then $\boldsymbol{\Phi}(u_0)=\mathbf{N}\boldsymbol{\Psi}(u_0)$ and so u_0 satisfies the first boundary condition (17). We will show, using (27) and (25), that u_0 satisfies the second boundary condition (17)

$$\boldsymbol{\Phi}(Au_0)-\mathbf{D}\mathbf{F}(Au_0)-\mathbf{N}\boldsymbol{\Psi}(Au_0)=\mathbf{N}\mathbf{c}_2+\mathbf{D}\mathbf{c}_3-$$
$$-\mathbf{D}\mathbf{F}(\mathbf{z})(\mathbf{N}\mathbf{c}_2+\mathbf{D}\mathbf{c}_3)-\mathbf{D}\mathbf{F}\left(\hat{A}^{-1}\mathbf{q}\right)\mathbf{c}_3-\mathbf{D}\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4-$$
$$-\mathbf{N}\boldsymbol{\Psi}(\mathbf{z})(\mathbf{N}\mathbf{c}_2+\mathbf{D}\mathbf{c}_3)-\mathbf{N}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{q}\right)\mathbf{c}_3-\mathbf{N}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4=$$
$$=\mathbf{N}\left[\mathbf{V}\mathbf{c}_2-\mathbf{K}_2\mathbf{c}_3-\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4\right]+$$
$$+\mathbf{D}\left[-\mathbf{F}(\mathbf{z})\mathbf{N}\mathbf{c}_2+\mathbf{K}_1\mathbf{c}_3-\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4\right]=\mathbf{N}\mathbf{0}+\mathbf{D}\mathbf{0}=\mathbf{0},$$

where $\mathbf{K}_1$, $\mathbf{K}_2$ from (19). So $u_0\in\mathfrak{D}(B_1)$. Now, using (25) and (28) we will show that $u_0\in\ker B_1$

$$B_1u_0=A^2u_0-\mathbf{q}\mathbf{F}(Au_0)-\mathbf{g}\mathbf{F}\left(A^2u_0\right)=\mathbf{q}\mathbf{c}_3+\mathbf{g}\mathbf{c}_4-$$
$$-\mathbf{q}\left[\mathbf{F}(\mathbf{z})(\mathbf{N}\mathbf{c}_2+\mathbf{D}\mathbf{c}_3)+\mathbf{F}\left(\hat{A}^{-1}\mathbf{q}\right)\mathbf{c}_3+\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4\right]-$$
$$-\mathbf{g}\mathbf{F}(\mathbf{q})\mathbf{c}_3-\mathbf{g}\mathbf{F}(g)\mathbf{c}_4=$$
$$=\mathbf{q}\left[-\mathbf{F}(\mathbf{z})\mathbf{N}\mathbf{c}_2+\mathbf{K}_1\mathbf{c}_3-\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{c}_4\right]+$$
$$+\mathbf{g}\left[-\mathbf{F}(\mathbf{q})\mathbf{c}_3+\mathbf{W}\mathbf{c}_4\right]=\mathbf{q}\mathbf{0}+\mathbf{g}\mathbf{0}=0.$$

So there exists a nonzero element $u_0\in\mathfrak{D}(B_1)$ and $u_0\in\ker B_1$. This means that B_1 is not injective. Hence the operator B_1 is injective if and only if $\det\mathbf{L}\neq\mathbf{0}$.

(ii) Since $\det\mathbf{L}\neq\mathbf{0}$, the system (24) for all $f\in X$ has an unique solution

$$col\left(\boldsymbol{\Psi}(u),\ \boldsymbol{\Psi}(Au),\ \mathbf{F}(Au),\ \mathbf{F}\left(A^2u\right)\right)=$$
$$=\mathbf{L}^{-1}col\left(\mathbf{F}\left(\hat{A}^{-1}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-2}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-1}f\right),\ \mathbf{F}(f)\right)\quad(30)$$

and the operator B_1, by statement (i), is injective. Substituting (30) into (23) we obtain the unique solution (20) of the problem (16), (17). In the above solution an element f is arbitrary. Consequently, $R(B_1)=X$. Since the operators $\hat{A}^{-2}$, $\hat{A}^{-1}$ and the functional vectors $\mathbf{F}$ and $\boldsymbol{\Psi}$ are bounded, from (20) follows the boundedness of B_1^{-1}, i. e. the operator B_1 is correct. The theorem is proved.

The next corollary follows from the above theorem for $\mathbf{q}=\mathbf{g}=\mathbf{0}$ and is useful for solving some classes of differential equations with nonlocal boundary conditions.

Corollary 2. Let the operators A, $\hat{A}$, the vectors $\mathbf{z}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, $\mathbf{F}$, $\mathbf{V}$ and matrices $\mathbf{D}$, $\mathbf{N}$ be defined as in Theorem 2 and the operator $B_1: X\to X$ be defined by

$$B_1u=A^2u=f,$$
$$\mathfrak{D}(B_1)=\{u\in\mathfrak{D}(A^2):\boldsymbol{\Phi}(u)=\mathbf{N}\boldsymbol{\Psi}(u),\quad(31)$$
$$\boldsymbol{\Phi}(Au)=\mathbf{D}\mathbf{F}(Au)+\mathbf{N}\boldsymbol{\Psi}(Au)\}.$$

Then:

(i) The operator B_1 corresponding to the problem (31) is injective if and only if

$$\det\mathbf{L}_1=\det\begin{pmatrix}\mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{I}_n-\mathbf{F}(\mathbf{z})\mathbf{D}\\ \mathbf{V} & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N} & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{D}\\ \mathbf{0}_n & \mathbf{V} & -\boldsymbol{\Psi}(\mathbf{z})\mathbf{D}\end{pmatrix}\neq0.\quad(32)$$

(ii) If the operator B_1 is injective, then it is correct and the unique solution of (31) is given by

$$u=B_1^{-1}f=\hat{A}^{-2}f+\left(\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{D}\right)\times$$
$$\times\mathbf{L}_1^{-1}col\left(\mathbf{F}\left(\hat{A}^{-1}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-2}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-1}f\right)\right).\quad(33)$$

Proof: (i) For $\mathbf{g}=\mathbf{q}=\mathbf{0}$ from (18) and (19) immediately follows

$$\det\mathbf{L}=\det\begin{pmatrix}\mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{I}_n-\mathbf{F}(\mathbf{z})\mathbf{D} & \mathbf{0}_n\\ \mathbf{V} & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N} & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{D} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{V} & -\boldsymbol{\Psi}(\mathbf{z})\mathbf{D} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \mathbf{I}_n\end{pmatrix}.\quad(34)$$

It is evident that $\det\mathbf{L}=\det\mathbf{L}_1$. From (20) for $\mathbf{g}=\mathbf{q}=\mathbf{0}$ follows the solution of (31)

$$u=B_1^{-1}f=\hat{A}^{-2}f+\left(\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{D},\ 0\right)\times$$
$$\times\mathbf{L}^{-1}col\left(\mathbf{F}\left(\hat{A}^{-1}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-2}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-1}f\right),\ \mathbf{F}(f)\right).\quad(35)$$

It is easy to verify that

$$\left(\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{D},\ 0\right)\times$$
$$\times\mathbf{L}^{-1}col\left(\mathbf{F}\left(\hat{A}^{-1}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-2}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-1}f\right),\ \mathbf{F}(f)\right)=$$
$$=\left(\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{N},\ \hat{A}^{-1}\mathbf{z}\mathbf{D}\right)\times$$
$$\times\mathbf{L}_1^{-1}col\left(\mathbf{F}\left(\hat{A}^{-1}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-2}f\right),\ \boldsymbol{\Psi}\left(\hat{A}^{-1}f\right)\right).$$

Hence, from (35) follows (33).

Examples

In the next example we use the extension method from Theorem 1.

Example. The multipoint problem for loaded integro-differential equation on $\mathbb{C}\,[0,1]$

$$u''-3t\int_0^1 x^2u''(x)\mathrm{d}x+\frac{1}{2}\left(t^2+1\right)\left[u'(1)-u'(0)\right]=$$
$$=8t^2+2t+12, \quad (36)$$
$$u(0)=\frac{1}{6}u(1/2)+\frac{1}{18}u(1),\ u'(0)=\frac{2}{9}u(1)$$

is correct and the unique solution of (36) is given by

$$u(t) = 4t^3 + 2t^2 + 2t + 1. \quad (37)$$

Proof: If we compare (36) with (10), it is natural to take $Au = u''(t)$, $\mathfrak{D}(A) = \{u \in \mathbb{C}^2[0, 1]\}$, $X_A^2=\mathbb{C}^2[0,1]$, $X_A^1=\mathbb{C}^1[0,1]$, $m=n=2$, $\mathbf{z}=(z_1, z_2)=(1, t)$, $\hat{A}u=Au$,

$$\mathfrak{D}(\hat{A})=\{u\in\mathfrak{D}(A): u(0)=u'(0)=0\},$$
$$Bu=u''-3t\int_0^1 x^2u''(x)\mathrm{d}x+\frac{1}{2}\left(t^2+1\right)\left[u'(1)-u'(0)\right]=$$
$$=u''-3t\int_0^1 x^2u''(x)\mathrm{d}x+\frac{1}{2}\left(t^2+1\right)\int_0^1 u''(x)\mathrm{d}x,$$
$$\mathfrak{D}(B)=$$
$$=\left\{u(x)\in\mathfrak{D}(A): \begin{pmatrix}u(0)\\u'(0)\end{pmatrix}=\begin{pmatrix}1/6 & 1/18\\0 & 2/9\end{pmatrix}\begin{pmatrix}u(1/2)\\u(1)\end{pmatrix}\right\}. \quad (38)$$

Since (5), the operator $\hat{A}$, is correct and its solution is $\hat{A}^{-1}f(t)=\int_0^t(t-x)f(x)\mathrm{d}x$. Further comparing (36), (38) with (10), we take $g_1=3t$, $g_2=-\frac{1}{2}\left(t^2+1\right)$, $f = 8t^2 + 2t + 12$,

$$\mathbf{N}=\begin{pmatrix}1/6 & 1/18\\0 & 2/9\end{pmatrix},\ F_1(Au)=\int_0^1 x^2u''(x)\mathrm{d}x,$$
$$F_2(Au)=\int_0^1 u''(x)\mathrm{d}x.$$

Then

$$F_1(f)=\int_0^1 x^2f(x)\mathrm{d}x,\ F_2(f)=\int_0^1 f(x)\mathrm{d}x,$$
$$\mathbf{\Phi}(u)=\begin{pmatrix}\Phi_1(u)\\\Phi_2(u)\end{pmatrix}=\begin{pmatrix}u(0)\\u'(0)\end{pmatrix},$$
$$\mathbf{\Psi}(u)=\begin{pmatrix}\Psi_1(u)\\\Psi_2(u)\end{pmatrix}=\begin{pmatrix}u(1/2)\\u(1)\end{pmatrix}.$$

The set $\mathbf{z} = (1, t)$ is biorthogonal to (Φ_1, Φ_2). From $|\Psi_1(u)| = |u(1/2)| \le \|u\|_{\mathbb{C}} + \|u'\|_{\mathbb{C}} = \|u\|_{\mathbb{C}^1}$ follows that $\Psi_1\in\mathbb{C}^{1*}=X_A^{m-1*}=X_A^{1*}$. By analogy Ψ_2, $\Psi_i \in \mathbb{C}^{1*}$, $i=1,\ 2$. Further from $\left|F_1(f)\right|=\left|\int_0^1 x^2f(x)\mathrm{d}x\right|\le\|f\|_{\mathbb{C}}$ it follows that $F_1\in\mathbb{C}[0,1]^*=X^*$. By analogy it is proved that $F_2\in X^*$. We can apply Theorem 1. Now we calculate

$$\hat{A}^{-1}g_1(t)=\int_0^t(t-x)g_1(x)\mathrm{d}x=\int_0^t(t-x)3x\,\mathrm{d}x=\frac{t^3}{2},$$
$$\hat{A}^{-1}g_2(t)=-\frac{1}{2}\int_0^t(t-x)\left(x^2+1\right)\mathrm{d}x=-\frac{t^2\left(t^2+6\right)}{24}.$$

Compute

$$\hat{A}^{-1}\mathbf{g}=\left(\hat{A}^{-1}g_1,\hat{A}^{-1}g_2\right)=\left(\frac{t^3}{2},\ -\frac{t^2\left(t^2+6\right)}{24}\right).$$

Further we find $\Psi_1(z_1) = z_1(1/2) = 1$, $\Psi_1(z_2) = z_2(1/2) = 1/2$, $\Psi_2(z_1) = z_1(1) = 1$, $\Psi_2(z_2) = z_2(1) = 1$. Then $\mathbf{\Psi}(\mathbf{z})=\begin{pmatrix}1 & 1/2\\1 & 1\end{pmatrix}$. Further compute $\Psi_1\left(\hat{A}^{-1}g_1\right)=1/16$, $\Psi_2\left(\hat{A}^{-1}g_1\right)=1/2$, $\Psi_1\left(\hat{A}^{-1}g_2\right)=-25/384$, $\Psi_2\left(\hat{A}^{-1}g_2\right)=-7/24$, then

$$\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})=\begin{pmatrix}1/16 & -25/384\\1/2 & -7/24\end{pmatrix}.$$

Now we find

$$F_1(g_1)=\int_0^1 3x^3\,\mathrm{d}x=3/4,$$
$$F_1(g_2)=\frac{1}{2}\int_0^1 x^2\left(x^2+1\right)\mathrm{d}x=-4/15,$$
$$F_2(g_1)=\int_0^1 3x\,\mathrm{d}x=3/2,$$
$$F_2(g_2)=-\frac{1}{2}\int_0^1\left(x^2+1\right)\mathrm{d}x=-2/3.$$

Then

$$\mathbf{F}(g)=\begin{pmatrix}3/4 & -4/15\\3/2 & -2/3\end{pmatrix}.$$

Since

$$\mathbf{W}=\mathbf{I}_2-\mathbf{F}(\mathbf{g})=\begin{pmatrix}1/4 & 4/15\\-3/2 & 5/3\end{pmatrix},$$
$$\mathbf{V}=\mathbf{I}_2-\mathbf{\Psi}(\mathbf{z})\mathbf{N}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}1 & 1/2\\1 & 1\end{pmatrix}\begin{pmatrix}1/6 & 1/18\\0 & 2/9\end{pmatrix}=$$
$$=\begin{pmatrix}5/6 & -1/6\\-1/6 & 13/18\end{pmatrix},$$

and $\det\mathbf{W} \ne 0$, $\det\mathbf{V} \ne 0$, the problem (36), by Theorem 1 (ii), is correct. For $f = 8t^2 + 2t + 12$ we calculate

$$\hat{A}^{-1}f(t)=\int_0^t (t-x)(2x+4)\mathrm{d}x=\frac{t^2\left(2t^2+t+18\right)}{3},$$

$$F_1(f)=\int_0^1 x^2\left(8x^2+2x+12\right)\mathrm{d}x=61/10,$$

$$F_2(f)=\int_0^1\left(8x^2+2x+12\right)\mathrm{d}x=47/3.$$

Then $\mathbf{F}(f)=col(61/10,\,47/3)$. We also compute

$$\Psi_1(\hat{A}^{-1}f)=\hat{A}^{-1}f|_{t=1/2}=19/12,$$

$$\Psi_2(\hat{A}^{-1}f)=\hat{A}^{-1}f|_{t=1}=7.$$

Then

$$\boldsymbol{\Psi}\left(\hat{A}^{-1}f\right)=col\left(19/12,\,7\right).$$

Substitution of these values into (12) yields the solution to the problem (36)

$$u(t)=\hat{A}^{-1}f+[\hat{A}^{-1}\mathbf{g}+\mathbf{zNV}^{-1}\boldsymbol{\Psi}(A^{-1}\mathbf{g})]\mathbf{W}^{-1}\mathbf{F}(f)+$$
$$+\mathbf{zNV}^{-1}\boldsymbol{\Psi}(\hat{A}^{-1}f)=\frac{t^2\left(2t^2+t+18\right)}{3}+$$
$$+\left[\left(\frac{t^3}{2},\,-\frac{t^2\left(t^2+6\right)}{24}\right)+(1,\,t)\begin{pmatrix}1/6 & 1/18\\ 0 & 2/9\end{pmatrix}\times\right.$$
$$\left.\times\begin{pmatrix}5/6 & -1/6\\ -1/6 & 13/18\end{pmatrix}^{-1}\begin{pmatrix}1/16 & -25/384\\ 1/2 & -7/24\end{pmatrix}\right]\frac{1}{49}\begin{pmatrix}100 & -16\\ 90 & 15\end{pmatrix}\times$$
$$\times\begin{pmatrix}61/10\\ 47/3\end{pmatrix}+(1,\,t)\begin{pmatrix}1/6 & 1/18\\ 0 & 2/9\end{pmatrix}\begin{pmatrix}5/6 & -1/6\\ -1/6 & 13/18\end{pmatrix}^{-1}\times$$
$$\times\begin{pmatrix}19/12\\ 7\end{pmatrix}=4t^3+2t^2+2t+1.$$

Conclusion

The main results of this paper are Theorems 1 and 2, where the problems $Bu=f$, $B_1u=f$ are solved by extension method. This method is essentially simpler and more convenient in the case of quadratic operator $B_1=B^2$. In this case the solvability condition and a solution of $B_1u=f$ can be obtained by application of the formula for solution of $Bu=f$ twice. The upcoming Part 2 of this paper will be devoted to decomposition method for this case. Note that the extension method is a generalization of direct method which is presented in [30]. The essential ingredient in our approach is the extension of the main idea in [26].

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Метод нахождения точных решений для интегро-дифференциальных уравнений Фредгольма с многоточечными и интегральными краевыми условиями. Часть 1. Метод расширения

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Введение: краевые задачи для дифференциальных и интегро-дифференциальных уравнений с многоточечными и нелокальными граничными условиями возникают в различных областях механики, физики, биологии, биотехнологии, химической инженерии, медицинской науки, финансов и других. Нахождение точных решений краевых задач с фредгольмовыми интегро-дифференциальными уравнениями является трудной проблемой. В большинстве случаев решения получаются численными методами. **Цель:** поиск необходимых и достаточных условий разрешимости абстрактных операторных уравнений и метод построения их точных решений. **Результаты:** предложен прямой метод для точного решения некоторого класса обыкновенных дифференциальных или фредгольмовых интегро-дифференциальных уравнений с сепарабельными ядрами и многоточечными и интегральными граничными условиями. Исследованы абстрактные уравнения вида $Bu = Au - \mathbf{gF}(Au) = f$ и $B_1u = A^2u - \mathbf{qF}(Au) - \mathbf{gF}(A^2u) = f$ с нелокальными граничными условиями $\mathbf{\Phi}(u) = \mathbf{N\Psi}(u)$ и $\mathbf{\Phi}(u) = \mathbf{N\Psi}(u)$, $\mathbf{\Phi}(Au) = \mathbf{DF}(Au) + \mathbf{N\Psi}(Au)$ соответственно, где $\mathbf{q}$, $\mathbf{g}$ являются векторами, $\mathbf{D}$, $\mathbf{N}$ — матрицами, а $\mathbf{F}$, $\mathbf{\Phi}$, $\mathbf{\Psi}$ — функциональными векторами. Предложенный метод прост в использовании и может быть легко интегрирован в любую систему компьютерной алгебры. Исследована корректность уравнений вида $Bu = f$ и $B_1u = f$ и их точные решения. Вторая часть этой статьи будет посвящена случаю, когда оператор B_1 имеет квадратичную факторизацию.

Ключевые слова — дифференциальные и фредгольмовы интегро-дифференциальные уравнения, многоточечные и нелокальные интегральные граничные условия, разложение операторов, корректность операторов, точные решения.

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