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Articles

Decomposition of abstract linear operators on Banach spaces

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Introduction: *The majority of the known decomposition methods for solving boundary value problems (Adomian decomposition method, natural transform decomposition method, modified Adomian decomposition method, combined Laplace transform — Adomian decomposition method, and Domain decomposition method) use so-called Adomian polynomials or iterations to get approximate solutions. To our knowledge, a direct method for obtaining an exact analytical solution is not yet proposed.* ***Purpose:*** *Developing, in an arbitrary Banach space, a new universal decomposition method for the class of ordinary or partial integro-differential equations with non-local and initial boundary conditions in terms of the abstract operator equation $B_1x = f$.* ***Results:*** *A class of integro-differential equations in a Banach space with non-local and initial boundary conditions in terms of an abstract operator equation $B_1x = \mathcal{A}x - \mathbf{S}_0\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) = f$, $x = D(B_1)$has been studied, where $\mathcal{A}$, A are linear abstract operators, $\mathbf{S}_0$, $\mathbf{G}_0$ are vectors and $\mathbf{\Phi}$, $\mathbf{F}$ the functional vectors. Usually, $\mathcal{A}$, A are linear ordinary or partial differential operators, and $\mathbf{F}(Ax)$, $\mathbf{\Phi}(Ax)$ are Fredholm integrals. The existence and uniqueness are proved under the assumption that the operator B_1 has a decomposition of the form $B_1 = B_0B$ with B and B_0 being different abstract linear operators of special forms. The proposed decomposition method is universal and essentially different from other decomposition methods in the relevant literature. This method can be applied to either ordinary integro-differential or partial integro-differential equations, providing a unique exact solution in closed analytical form in a Banach space. The stages of the method are illustrated by numerical examples corresponding to specific problems. Computer algebra system Mathematica is used to demonstrate the solution outcomes and to assess the effectiveness of the analysis.* ***Practical relevance:*** *The main advantage of the proposed solution method is that it can be integrated in the interface of any CAS software in an easy, programing-free way.*

Keywords *— correct operator, decomposition (factorization) of operators (equations), integro-differential equations, boundary value problems, exact solution.*

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Preliminaries and auxiliary results

Integro-differential equations are used in many problems from science and engineering. The integro-differential operators describing these problems are complicated and the exact solution of the corresponding boundary value problems is a difficult task. In some cases, the boundary value problem (BVP) can be transformed into a simpler one involving simpler operators and thus the solution can be found easier.

The decomposition (factorization) methods were used in many applications in gas dynamics, transport theory, electromagnetism, quantum physics, mechanics, hydrodynamics and cosmology [1–14]. In pure mathematics, decomposition (factorization) method continues to be a very successful tool for solving variational inequalities, linear and nonlinear ordinary and partial differential and Volterra — Fredholm integro-differential equations as well as systems of partial differential equations. This method is very important for solving fuzzy Volterra — Fredholm integral equations, integro-differential equations of fractional order and delay differential equations [15–32]. However, almost all the approaches of the literature listed above do not give exact solutions in their closed analytical forms and the corresponding problems are not formulated in terms of abstract operator equations. Thus, the decomposition methods proposed and employed in these problems are not universal.

Exact solutions in their analytical form for abstract operator equations in Hilbert and Banach spaces were obtained by quadratic and biquadratic decompositions of the integro-differential equations in [33–37]. The universal decomposition method for the abstract linear operator equation

$$B_1x = A^2x - \mathbf{SF}(Ax) - \mathbf{GF}(A^2x) = f,\ x \in D(B_1)$$

was given in [38] on a Hilbert space. We note that Banach spaces play a central role in functional analysis and it is important to study the exact solutions of correct BVPs in the context of Banach spaces. This work is a natural continuation of [38] to a Banach space and introduces the universal decomposition method for the similar linear abstract operator equation

$$B_1x = \mathcal{A}x - \mathbf{S}_0\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) = f,\ x \in D(B_1), \quad (1)$$

where $\mathcal{A}$, A are linear abstract operators; $\mathbf{S}_0$, $\mathbf{G}_0$ are vectors and $\mathbf{\Phi}$, $\mathbf{F}$ — functional vectors. The decomposition method proposed here is different than the well-known decomposition methods (namely the Adomian decomposition method, the natural transform decomposition method, modified Adomian decomposition method, the combined Laplace transform — Adomian decomposition method and domain decomposition method). In the relevant literature, the so-called Adomian polynomials or iterations were used to obtain numerical solutions (see [1–32]). The class of integro-differential equations with nonlocal boundary conditions described by an abstract operator equation is studied in [39], where all calculations are reproducible in any program of symbolic calculations and the computer codes in Mathematica are given.

In the sections that follow we use the following notations, definitions and statements.

We denote by X a complex Banach space and by X^* the adjoint space of X, i. e. the set of all complex-valued linear and bounded functionals f on X. We denote by $f(x)$ the value of f on x.

We write $D(A)$ and $R(A)$ for the domain and the range of the operator A, respectively. An operator A: $X \to X$ is called *correct* if $R(A) = X$ and the inverse A^{-1} exists and is continuous on X. If for an operator B_1 there are two operators B_0, B such that B_1 can be written as a product $B_1 = B_0B$, then we say that B_0B is a decomposition (factorization) of B_1 and write $B_1 = B_0B$. An operator B_1: $X \to X$ is called *quadratic* (*biquadratic*) if there exists an operator B: $X \to X$ such that $B_1 = B^2$, ($B_1 = B^4$) and the corresponding decomposition $B_1 = B^2$, ($B_1 = B^4$) is called *quadratic* (*biquadratic*). Recall that the problem $Ax = f$ is called *correct*, if the operator A is correct. If x, $g_i \in X$ and $\Phi_i \in X^*$, $i = 1, \ldots, m$ then we denote by $\mathbf{g} = (g_1, \ldots, g_m)$, $\mathbf{\Phi} = \mathrm{col}(\Phi_1, \ldots, \Phi_m)$ and $\mathbf{\Phi}(x) = \mathrm{col}(\Phi_1(x), \ldots, \Phi_m(x))$ and we write $\mathbf{g} \in X^m$, $\mathbf{\Phi} \in X^m$. We will denote by $\mathbf{\Phi}(\mathbf{g})$ the $m \times m$ matrix whose i, j-th entry $\Phi_i(g_j)$ is the value of functional Φ_i on element g_j. Note that $\mathbf{\Phi}(\mathbf{gC}) = \mathbf{\Phi}(\mathbf{g})\mathbf{C}$, where $\mathbf{C}$ is a $m \times k$ constant matrix. We will also denote by $\mathbf{0}_m$ and $\mathbf{I}_m$ the zero and identity $m \times m$ matrices.

Next, we state some useful outcomes. Specifically, Theorem 1 from [40] and Corollary 3.11 from [33].

Theorem 1. Let X, Y and Z be Banach spaces and A_0: $X \to Y$ be a correct operator with $D(A_0) \subset Z \subseteq X$. Further let the vector $\mathbf{G}_0 = \left(g_1^{(0)}, \ldots, g_m^{(0)}\right) \in Y^m$ and the column vector $\mathbf{\Phi} = \mathrm{col}(\phi_1, \ldots, \phi_m)$, where $\phi_1, \ldots, \phi_m \in Z^*$ and their restrictions on $D(A_0)$ are linearly independent. Then:

(i) The operator B_0: $X \to X$ defined by

$$B_0x = A_0x - \mathbf{G}_0\mathbf{\Phi}(x) = f,\ D(B_0) = D(A_0),\ f \in X, \quad (2)$$

is correct if and only if

$$\det \mathbf{L}_0 = \det\left[\mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\right] \neq 0. \quad (3)$$

(ii) If B_0 is correct, then for any $f \in Y$, the unique solution of (2) is given by

$$x = B_0^{-1}f = A_0^{-1}f + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right). \quad (4)$$

Corollary 1. Let A be a correct operator on a Banach space X and the components of the vectors $\mathbf{G} = (g_1, \ldots, g_m)$, $\mathbf{F} = \mathrm{col}(F_1, \ldots, F_m)$ are arbitrary elements of X and X^*, respectively. Then the operator B: $X \to X$ defined by

$$Bx = Ax - \mathbf{GF}(Ax) = f,\ D(B) = D(A),\ f \in X \quad (5)$$

is correct if and only if

$$\det \mathbf{L} = \det[\mathbf{I}_m - \mathbf{F}(\mathbf{G})] \neq 0. \quad (6)$$

If B is correct, then the unique solution of (5) for every $f \in X$ is given by

$$X = B^{-1}f = A^{-1}f + A^{-1}\mathbf{G}\mathbf{L}^{-1}\mathbf{F}(f). \quad (7)$$

Decomposition of abstract linear operators on a Banach space

In this section we investigate problem (1) where B_1 is not quadratic but it can be written as a product of two other correct operators B_0, B i. e. $B_1 = B_0B$. In this case the solvability condition and the solution formulation are essentially simpler than in the general case.

We will prove the following theorem using the technique that was first applied for the case of Hilbert space in Theorem 2.5 [38], where a given operator B_0 of the type $B_0x = A_0x - \mathbf{G}_0\mathbf{\Phi}(A_0x) = f$, $x \in D(B_0)$ and an operator A is densely defined. We use a different operator B_0 without the assumption of density of $D(A)$ on X.

Theorem 2. Let X and Z be Banach spaces, $Z \subseteq X$, the vectors $\mathbf{G} = (g_1, \ldots, g_m)$, $\mathbf{G}_0 = \left(g_1^{(0)}, \ldots, g_m^{(0)}\right)$, $\mathbf{S}_0 = \left(s_1^{(0)}, \ldots, s_m^{(0)}\right) \in X^m$, the components of the vectors $\mathbf{F} = \mathrm{col}(F_1, \ldots, F_m)$ and $\mathbf{\Phi} = \mathrm{col}(\Phi_1, \ldots, \Phi_m)$ belong to X^* and Z^*, respectively, and the operators B_0, B, B_1: $X \to X$ defined by

$$B_0x = A_0x - \mathbf{G}_0\mathbf{\Phi}(x) = f,\ D(B_0) = D(A_0) \subset Z; \quad (8)$$

$$Bx = Ax - \mathbf{GF}(Ax) = f,\ D(B) = D(A); \quad (9)$$

$$B_1x = A_0Ax - \mathbf{S}_0\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) = f,\quad D(B_1) = D(A_0A), \quad (10)$$

where A_0 and A are linear correct operators on X; $\mathbf{G} \in D(A_0)^m$ and the restrictions of $\Phi_1, \ldots, \Phi_m$ on $D(A_0)$ are linearly independent. Then the following statements are satisfied:

(i) If

$$\mathbf{S}_0 \in R(B_0)^m \text{ and } \mathbf{S}_0 = B_0\mathbf{G} = = A_0\mathbf{G} - \mathbf{G}_0\mathbf{\Phi}(\mathbf{G}), \quad (11)$$

then the operator B_1 can be decomposed in $B_1 = B_0B$.

(ii) If in addition the components of the vector $\mathbf{F} = \mathrm{col}(F_1, ..., F_m)$ are linearly independent elements of X^* and since the operator B_1 can be decomposed in $B_1 = B_0B$, then (11) is fulfilled.

(iii) If the operator B_1 can be decomposed in $B_1 = B_0B$ then B_1 is correct if and only if the operators B_0, and B are correct which means that

$$\det\mathbf{L}_0 = \det\left[\mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\right] \neq 0 \text{ and}$$

$$\det\mathbf{L} = \det[\mathbf{I}_m - \mathbf{F}(\mathbf{G})] \neq 0. \quad (12)$$

(iv) If the operator B_1 has the decomposition in $B_1 = B_0B$ and is correct, then the unique solution of (10) is

$$x = B_1^{-1}f = A^{-1}A_0^{-1}f + A^{-1}\mathbf{G}\mathbf{L}^{-1}\mathbf{F}\left(A_0^{-1}f\right) + \left[A^{-1}A_0^{-1}\mathbf{G}_0 + A^{-1}\mathbf{G}\mathbf{L}^{-1}\mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right)\right]\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right). \quad (13)$$

Proof: (i) Taking into account that $G \in D(A_0)^m$ and (8)–(10) we get

$$D(B_0B) = \{x \in D(B): Bx \in D(B_0)\} =$$
$$= \{x \in D(A): Ax - \mathbf{GF}(Ax) \in D(A_0)\} =$$
$$= \{x \in D(A): Ax \in D(A_0)\} = D(A_0A) = D(B_1).$$

So $D(B_1) = D(B_0B)$. Let $y = \mathrm{B}x$. Then for each $x \in D(A_0A)$ and taking into account (8) and (9) we have

$$B_0Bx = B_0y = A_0y - \mathbf{G}_0\mathbf{\Phi}(y) =$$
$$= A_0[Ax - \mathbf{GF}(Ax)] - \mathbf{G}_0\mathbf{\Phi}(Ax - \mathbf{GF}(Ax)) =$$
$$= A_0Ax - A_0\mathbf{GF}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) + \mathbf{G}_0\mathbf{\Phi}(\mathbf{G})\mathbf{F}(Ax) =$$
$$= A_0Ax - \mathbf{G}_0\mathbf{\Phi}(Ax) - [A_0\mathbf{G} - \mathbf{G}_0\mathbf{\Phi}(\mathbf{G})]\mathbf{F}(Ax) =$$
$$= A_0Ax - B_0\mathbf{GF}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax), \quad (14)$$

where the relation $B_0\mathbf{G} = A_0\mathbf{G} - \mathbf{G}_0\mathbf{\Phi}(\mathbf{G})$ results naturally from (8) by substituting $x = \mathbf{G}$.

By comparing (14) with (10), it is easy to verify that $B_1x = B_0Bx$ for each $x \in D(A_0A)$ if a vector $\mathbf{S}_0$ satisfies (11).

(ii) Let the operator B_1 can be decomposed in $B_1 = B_0B$. Then by comparing (14) with (10) we obtain

$$(B_0\mathbf{G} - \mathbf{S}_0)\mathbf{F}(Ax) = 0. \quad (15)$$

Because of the correctness of operators A, A_0 and the linear independence of $F_1, ..., F_m$, there exists a system $x_1, ..., x_m \in D(A_0A)$ such that $\mathbf{F}(A_0x_0) = \mathbf{I}_m$ where $\mathbf{x}_0 = (x_1, ..., x_m)$. By substituting $x = \mathbf{x}_0$ into (15) we get $\mathbf{S}_0 = B_0\mathbf{G}$. Hence $\mathbf{S}_0 \in R(B_0)^m$ and $\mathbf{S}_0 = B_0\mathbf{G} = A_0\mathbf{G} - \mathbf{G}_0\mathbf{\Phi}(\mathbf{G})$.

(iii) Let the operator B_1 be defined by (10) where $\mathbf{S}_0 = B_0\mathbf{G}$. Then equation (10) can be equivalently represented as a matrix equation:

$$B_1x = A_0Ax - \left(B_0\mathbf{G}, \mathbf{G}_0\right)\begin{pmatrix}\mathbf{F}\left(A_0^{-1}A_0Ax\right)\\ \mathbf{\Phi}\left(A_0^{-1}A_0Ax\right)\end{pmatrix} = f, \quad (16)$$

or

$$B_1 = \mathcal{A}x - \tilde{\mathbf{G}}\tilde{\mathbf{F}}(\mathcal{A}x) = f,\ D(B_1) = D(\mathcal{A}), \quad (17)$$

where

$$\mathcal{A} = AA_0;\ \tilde{\mathbf{G}} = (B_0\mathbf{G}, \mathbf{G}_0);$$

$$\tilde{\mathbf{F}} = \mathrm{col}\left(\hat{\mathbf{F}}, \hat{\mathbf{\Phi}}\right),\ \tilde{\mathbf{F}}(\mathcal{A}x) = \begin{pmatrix}\hat{\mathbf{F}}(\mathcal{A}x)\\ \hat{\mathbf{\Phi}}(\mathcal{A}x)\end{pmatrix},$$

then

$$\tilde{\mathbf{F}}(v) = \begin{pmatrix}\hat{\mathbf{F}}(v)\\ \hat{\mathbf{\Phi}}(v)\end{pmatrix} = \begin{pmatrix}\mathbf{F}\left(A_0^{-1}v\right)\\ \hat{\mathbf{\Phi}}\left(A_0^{-1}v\right)\end{pmatrix}.$$

Notice that the operator $\mathcal{A} = AA_0$ is correct, because of A and A_0 are correct operators, and the functional vector $\tilde{\mathbf{F}}$ is bounded, since the vectors $\hat{\mathbf{F}}, \hat{\mathbf{\Phi}}$ are bounded as a superposition of a bounded functional $\mathbf{F}, \mathbf{\Phi}$ respectively and a bounded operator A_0^{-1}. Then we apply Corollary 1. By this corollary the operator B_1 is correct if and only if

$$\det\mathbf{L}_1 = \det\left[\mathbf{I}_{2m} - \mathbf{F}\left(\tilde{\mathbf{G}}\right)\right] =$$
$$= \det\left[\begin{pmatrix}\mathbf{I}_m & 0_m\\ 0_m & \mathbf{I}_m\end{pmatrix} - \begin{pmatrix}\hat{\mathbf{F}}(B_0\mathbf{G}) & \hat{\mathbf{F}}(\mathbf{G}_0)\\ \hat{\mathbf{\Phi}}(B_0\mathbf{G}) & \hat{\mathbf{\Phi}}(\mathbf{G}_0)\end{pmatrix}\right] =$$
$$= \det\begin{pmatrix}\mathbf{I}_m - \mathbf{F}\left(\mathbf{G} - A_0^{-1}\mathbf{G}_0\mathbf{\Phi}(\mathbf{G})\right) & -\mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right)\\ -\mathbf{\Phi}\left(\mathbf{G} - A_0^{-1}\mathbf{G}_0\mathbf{\Phi}(\mathbf{G})\right) & \mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\end{pmatrix} =$$
$$= \det\begin{pmatrix}\mathbf{I}_m - \mathbf{F}(\mathbf{G}) + \mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right)\mathbf{\Phi}(\mathbf{G}) & -\mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right)\\ -\mathbf{\Phi}(\mathbf{G}) + \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\mathbf{\Phi}(\mathbf{G}) & \mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\end{pmatrix} \neq 0.$$

According to properties of determinants of matrices (Remark 1, [34]), taking $\mathbf{L}_1$ in the last formulation from above and adding $\mathbf{\Phi}(\mathbf{G})$ times the second column of $\mathbf{L}_1$ to its first column, the determinant is unchanged. We then get

$$\det\mathbf{L}_1 = \det\begin{pmatrix} \mathbf{I}_m - \mathbf{F}(\mathbf{G}) & -\mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right) \\ \mathbf{0}_m & \mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right) \end{pmatrix} =$$
$$= \det\left[\mathbf{I}_m - \mathbf{F}(\mathbf{G})\right]\det\left[\mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\right] =$$
$$= \det\mathbf{L}_0 \det\mathbf{L} \neq 0.$$

So we proved that the operator B_1 is correct if and only if (12) is fulfilled.

(iv) Let $x \in D(A_0A)$ and $B_0Bx = f$. Then by Theorem 1 (ii) since B_0, B are correct operators, we obtain

$$Bx = B_0^{-1}f = A_0^{-1}f + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right),$$
$$x = B^{-1}\left(A_0^{-1}f + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right)\right).$$

In the last equation we denote by $g = A_0^{-1}f + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right)$. Following strictly Corollary 1 (ii), we get

$$x = B^{-1}g = A^{-1}g + A^{-1}\mathbf{G}\mathbf{L}^{-1}\mathbf{F}(g) =$$
$$= A^{-1}\left(A_0^{-1}f + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right)\right) +$$
$$+ A^{-1}\mathbf{G}\mathbf{L}^{-1}\mathbf{F}\left(A_0^{-1}f + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right)\right) =$$
$$= A^{-1}A_0^{-1}f + A^{-1}A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right) +$$
$$+ A^{-1}\mathbf{G}\mathbf{L}^{-1}\left[\mathbf{F}\left(A_0^{-1}f\right) + \mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right)\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}f\right)\right],$$

which implies (13). Thus, the theorem has been proved.

The next theorem is useful for applications.

Theorem 3. Let X and Z be Banach spaces, $Z \subseteq X$ the vectors $\mathbf{G}_0 = \left(g_1^{(0)}, \ldots, g_1^{(0)}\right)$, $\mathbf{S}_0 = \left(s_1^{(0)}, \ldots, s_1^{(0)}\right) \in X^m$, the components of the vectors $\mathbf{F} = \mathrm{col}(F_1, \ldots, F_m)$ and $\mathbf{\Phi} = \mathrm{col}(\Phi_1, \ldots, \Phi_m)$ belong to X^* and Z^*, respectively, the operators $\mathcal{A}$, A, B_1: $X \to X$ and the operator B_1 defined by

$$B_1x = \mathcal{A}x - \mathbf{S}_0\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) = f,\ x \in D(B_1), \quad (18)$$

where A is a correct m-order differential operator and $\mathcal{A}$ is a n-order differential operator, $m < n$. Then the next statements are fulfilled:

(i) If there exist a bijective $n - m$ order differential operator A_0: $X \to X$ and the vector $\mathbf{G}$ such that

$$\mathcal{A} = A_0A,\ D(B_1) = D(A_0A),\ D(A_0) \subset Z; \quad (19)$$
$$\det\mathbf{L}_0 = \det\left[\mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\right] \neq 0; \quad (20)$$
$$\mathbf{G} = A_0^{-1}\mathbf{S}_0 + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right), \quad (21)$$

and the restrictions of $\Phi_1, \ldots, \Phi_m$ are linearly independent on $D(A_0)$, then the operator B_1 is decomposed in $B_1 = B_0B$, where B_0, B are given by (8), (9), respectively, the operator B_0 is constructed by the triple of elements A_0, $\mathbf{\Phi}$, $\mathbf{G}_0$ from (18)–(20), and the operator B by the operator A and vector $\mathbf{F}$ from (18) and the vector $\mathbf{G}$ from (21).

(ii) If in addition to (i) A_0 is correct, then B_1 is correct if and only if

$$\det\mathbf{L} = \det[\mathbf{I}_m - \mathbf{F}(\mathbf{G})] =$$
$$= \det\left[\mathbf{I}_m - \mathbf{F}\left(A_0^{-1}\mathbf{S}_0\right) - \mathbf{F}\left(A_0^{-1}\mathbf{G}_0\right) \times\right.$$
$$\left.\times \mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right)\right] \neq 0, \quad (22)$$

and the problem (18), (19) has the unique solution given by (13).

Proof: (i) If a bijective $n - m$ order differential operator A_0 and a vector $\mathbf{G}$ exist satisfying (19)–(21), then from (18) we get

$$B_1x = A_0Ax - \mathbf{S}_0\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) = f,$$
$$x \in D(A_0A). \quad (23)$$

From (23) we take the operator A and vector $\mathbf{F}$, whereas from (21) we take a vector $\mathbf{G}$ and construct the operator B according to the formula (9). To determine the operator B_0 by the formula (8), we take from (23) the operator A_0 and the vectors $\mathbf{\Phi}$, $\mathbf{G}_0$. We proved in the previous theorem (i) that $D(B_0B) = D(A_0A) = D(B_1)$. Substituting (21) into (8) we obtain

$$B_0\mathbf{G} = B_0\left[A_0^{-1}\mathbf{S}_0 + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right)\right] =$$
$$= A_0\left[A_0^{-1}\mathbf{S}_0 + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right)\right] -$$
$$- \mathbf{G}_0\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0 + A_0^{-1}\mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right)\right) =$$
$$= \mathbf{S}_0 + \mathbf{G}_0\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right) - \mathbf{G}_0\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right) -$$
$$- \mathbf{G}_0\mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right) =$$
$$= \mathbf{S}_0 + \mathbf{G}_0\left[\mathbf{I}_m - \mathbf{\Phi}\left(A_0^{-1}\mathbf{G}_0\right)\right] \times$$
$$\times \mathbf{L}_0^{-1}\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right) - \mathbf{G}_0\mathbf{\Phi}\left(A_0^{-1}\mathbf{S}_0\right) = \mathbf{S}_0.$$

$\mathbf{S}_0 = B_0\mathbf{G}$ and from (23) for $\mathbf{S}_0 = B_0\mathbf{G}$ and every $x \in D(B_1)$ we get

$$B_1x = A_0Ax - B_0\mathbf{G}\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) =$$
$$= B_0Ax - B_0\mathbf{G}\mathbf{F}(Ax) = B_0[Ax - \mathbf{G}\mathbf{F}(Ax)] = B_0Bx.$$

Thus we obtained the decomposition $B_1 = B_0B$.

(iii) If the statement (i) holds, then B_1 can be decomposed in $B_1 = B_0B$. By Theorem 3 (iii), B_1 is correct if and only if (12) holds or, taking into account (20) and (21), if and only if $\det\mathbf{L} = \det[\mathbf{I}_m - \mathbf{F(G)}] \neq 0$, or if and only if (22) is fulfilled. The last inequality immediately follows by substitution (21) into $\det\mathbf{L} = \det[\mathbf{I}_m - \mathbf{F(G)}]$. Since B_1 is correct and decomposed in $B_1 = B_0B$, by Theorem 2 (iv), we obtain the unique solution (13). So, the theorem is proved.

Remark. Usually as a Banach space X we have $C[a, b]$ or $L_p(a, b)$ and as a Banach space Z we have $C^k[a, b]$ or $W_p^k = (a, b)$, $k = 1, \ldots, n$.

Numerical examples

Let us examine several examples where our findings are applied and validated (the Mathematica notebook solving each example is available upon request).

Example 1. The operator B_1: $C[0, 1] \to C[0, 1]$ corresponding to the problem

$$x''(t) - t^2\int_0^1 t^3 x'(t)\mathrm{d}t - t\int_0^1 tx'(t)\mathrm{d}t = 2t + 1,$$

$$x(0) + x(1) = 0,\ x'(0) - 2x'(1) = 0 \qquad (24)$$

is correct. The unique solution of problem (24) is given by the formula

$$x(t) = -\frac{31990t^4 - 158464t^3 - 451860t^2 + 2502304t - 961985}{903720}. \qquad (25)$$

Proof: If we compare equation (24) with equations (18), (19), it is natural to denote $\mathbf{\Phi} = \Phi_1 = \Phi$, $\mathbf{F} = F_1 = F$, $\mathbf{G}_0 = g_1^{(0)} = G_0$, $\mathbf{S}_0 = s_1^{(0)} = S_0$, $\mathbf{I}_m = 1$, and to take $X = C[0, 1]$,

$$B_1x(t) = x''(t) - t^2\int_0^1 t^3x'(t)\mathrm{d}t - t\int_0^1 tx'(t)\mathrm{d}t = 2t + 1; \qquad (26)$$

$$D(B_1) = \{x(t) \in C^2[0, 1] : x(0) + x(1) = 0,\ x'(0) - 2x'(1) = 0\}; \qquad (27)$$

$$\mathcal{A}x = A_0Ax = x''(t); \qquad (28)$$

$$Ax(t) = x'(t),\ D(A) = \{x(t) \in C^1[0, 1] : x(0) = -x(1)\}; \qquad (29)$$

$$\Phi(Ax) = \int_0^1 tx'(t)\mathrm{d}t,\ \ F(Ax) = \int_0^1 t^3x'(t)\mathrm{d}t, \qquad (30)$$

$G_0 = t$, $S_0 = t^2$. Let us denote $Ax(t) = x'(t) = y(t) = y$. Then from (28) and (27) we have $y \in D(A_0)$, $A_0Ax = (x'(t))' = y'(t) = A_0y(t)$, $y(0) - 2y(1) = 0$. So we proved that

$$A_0y = y'(t),\ D(A_0) = \{y(t) \in C^1[0, 1]: y(0) - 2y(1) = 0\}.$$

Now we check the condition $D(B_1) = D(A_0A)$. By definition

$$D(A_0A) = \{x(t) \in D(A): Ax(t) \in D(A_0)\} = \{x(t) \in C^1[0, 1]: x(0) = -x'(1),\ x'(t) \in C^1[0, 1],\ x'(0) - 2x'(1) = 0\} = \{x(t) \in C^2[0, 1]: x(0) + x(1) = 0,\ x'(0) - 2x'(1) = 0\} = D(B_1).$$

So $D(B_1) = D(A_0A)$. It is easy to verify that the operators A_0, A are correct on $C[0, 1]$ and for every $f(t) \in C[0, 1]$ the following equations hold true

$$A_0^{-1}f(t) = \int_0^t f(s)\mathrm{d}s - 2\int_0^1 f(s)\mathrm{d}s; \qquad (31)$$

$$A_0^{-1}f(t) = \int_0^t f(s)\mathrm{d}s - \frac{1}{2}\int_0^1 f(s)\mathrm{d}s. \qquad (32)$$

From (30) we have

$$\Phi(f) = \int_0^1 sf(s)\mathrm{d}s,\ \ F(f) = \int_0^1 s^3f(s)\mathrm{d}s. \qquad (33)$$

It is evident that Φ, $F \in C^*[0, 1]$. Consequently, we can take $Z = C[0, 1] = X$.

Using (33) and (21) we find

$$F(S_0) = \int_0^1 s^3s^2\,\mathrm{d}s = \frac{1}{6},\ \ F(G_0) = \int_0^1 s^3s\,\mathrm{d}s = \frac{1}{5},$$

$$A_0^{-1}G_0 = \int_0^t s\,\mathrm{d}s - 2\int_0^1 s\,\mathrm{d}s = \frac{t^2}{3} - 1,$$

$$\Phi\left(A_0^{-1}G_0\right) = \int_0^1 s\left(\frac{s^2}{2} - 1\right)\mathrm{d}s = -\frac{3}{8},$$

$$A_0^{-1}S_0 = \int_0^t s^2\,\mathrm{d}s - 2\int_0^1 s^2\,\mathrm{d}s = \frac{t^3}{3} - \frac{2}{3},$$

$$\Phi\left(A_0^{-1}S_0\right) = \int_0^1 s\left(\frac{s^3}{2} - \frac{2}{3}\right)\mathrm{d}s = -\frac{4}{15},$$

$$L_0 = I_m - \Phi\left(A_0^{-1}G_0\right) = \frac{11}{8},\ \ L_0^{-1} = \frac{8}{11},$$

$$G = A_0^{-1}S_0 + A_0^{-1}G_0L_0^{-1}\Phi\left(A_0^{-1}S_0\right) = \frac{t^3}{3} - \frac{2}{3} + \left(\frac{t^2}{2} - 1\right)\frac{8}{11}\left(-\frac{4}{15}\right) = \frac{1}{165}\left(55t^3 - 16t^2 - 78\right).$$

Taking into account (33) we obtain

$$F(G)=\frac{1}{165}\int_0^1 s^3\left(55s^3-16s^2-78\right)\mathrm{d}s=-\frac{601}{6930}.$$

Since $\det L=\det[1-F(G)]=\dfrac{7531}{6930}\neq 0$ then $L^{-1}=\dfrac{6930}{7531}$, and by Theorem 3 (ii), problem (26), (27) or (24) is correct. By (32) we calculate

$$A^{-1}G=\frac{330t^4-128t^3-1872t+835}{3960},$$

$$A^{-1}A_0^{-1}G_0=\frac{t^3}{6}-t+\frac{5}{12}$$

and for $f(t)=2t+1$ by (31)–(33) we obtain

$$A_0^{-1}f=-4+t+t^2,\ \ A^{-1}A_0^{-1}f=\frac{19}{12}-4t+\frac{t^2}{2}+\frac{t^3}{3},$$

$$F\left(A_0^{-1}f\right)=-\frac{19}{30},\ \ \Phi\left(A_0^{-1}f\right)=-\frac{17}{12}.$$

Substituting these values into (13) we obtain the unique solution of (26), (27) or (24), which is given by (25).

Example 2. The operator B_1: $C[0,\pi]\to C[0,\pi]$ corresponding to the problem

$$x'''(t)-\sin t\int_0^{\pi}t^2x''(t)\mathrm{d}t-$$

$$-\cos t\int_0^{\pi/2}(t+1)x''(t)\mathrm{d}t=\sin 2t, \qquad (34)$$

$$x(0)+x(\pi)=0,\ x'(0)+3x'(\pi)=0,\ x''(0)+x''(\pi)=0,$$

is correct. The unique solution of the problem (34) is given by the formula

$$x(t)=\frac{1}{48}\Bigg[3\left(-2+\pi^2-6\pi t+4t^2+2\cos 2t\right)-\frac{\pi\left(2\pi^2-3\right)\left(8\cos t+\pi(\pi-2t-4\sin t)\right)}{\pi^3-2}+\frac{3(2+\pi)^2\left(4\left(\pi^2-4\right)\cos t-(2\pi-1)\times(\pi-2t-4\sin t)\right)}{2\left(\pi^3-2\right)}\Bigg]. \qquad (35)$$

Proof: If we compare (34) with equations (18), (19), it is natural to denote $\mathbf{\Phi}=\Phi_1=\Phi$, $\mathbf{F}=F_1=F$, $\mathbf{G}_0=g_1^{(0)}=G_0$, $\mathbf{S}_0=s_1^{(0)}=S_0$, $\mathbf{I}_m=1$, and to take $X=C[0,\pi]$,

$$B_1x(t)=x'''(t)-\sin t\int_0^{\pi}t^2x''(t)\mathrm{d}t-$$

$$-\cos t\int_0^{\pi/2}(t+1)x''(t)\mathrm{d}t=\sin 2t; \qquad (36)$$

$$D(B_1)=\{x(t)\in C^3[0,\pi]: x(0)+x(\pi)=0,$$
$$x'(0)+3x'(\pi)=0,\ x''(0)+x''(\pi)=0\}; \qquad (37)$$

$$\mathcal{A}x=A_0Ax=x'''(t); \qquad (38)$$

$$Ax(t)=x''(t);$$

$$D(A)=\{x(t)\in C^2[0,\pi]: x(0)=-x(\pi),$$
$$x'(0)+3x'(\pi)=0\}; \qquad (39)$$

$$\Phi(Ax)=\int_0^{\pi/2}(t+1)x''(t)\mathrm{d}t,$$

$$F(Ax)=\int_0^{\pi}t^2x''(t)\mathrm{d}t, \qquad (40)$$

$S_0=\sin t$, $G_0=\cos t$, $f=\sin 2t$. Denote $Ax(t)=x''(t)==y(t)=y$. Then from (37) and (38) we have $y\in D(A_0)$, $A_0Ax=(x''(t))'=y'(t)=A_0y(t)$, $y(0)+y(\pi)=0$. So we proved that

$$A_0y=y'(t),\ D(A_0)=$$
$$=\{y(t)\in C^1[0,\pi]: y(0)+y(\pi)=0\}. \qquad (41)$$

Now we check the condition $D(B_1)=D(A_0A)$. By definition

$$D(A_0A)=\{x(t)\in D(A): Ax(t)\in D(A_0)\}=$$
$$=\{x(t)\in C^2[0,\pi]: x(0)+x(\pi)=0,\ x'(0)+3x'(\pi)=0,$$
$$x''(t)\in C^1[0,\pi],\ x''(0)+x''(\pi)=0\}=$$
$$=\{x(t)\in C^3[0,\pi]: x(0)+x(\pi)=0,$$
$$x'(0)+3x'(\pi)=0,\ x''(0)+x''(\pi)=0\}=D(B_1).$$

So $D(B_1)=D(AA_0)$. It is easy to verify that the operators A, A_0 are correct on $C[0,\pi]$ and for every $f(t)\in C[0,\pi]$ from (39) and (41) follows that

$$A_0^{-1}f(t)=\int_0^t(t-s)f(s)\mathrm{d}s+$$
$$+\frac{1}{4}\int_0^{\pi}(2s-3t-\pi/2)f(s)\mathrm{d}s; \qquad (42)$$

$$A_0^{-1}f(t)=\int_0^t f(s)\mathrm{d}s-\frac{1}{2}\int_0^{\pi}f(s)\mathrm{d}s. \qquad (43)$$

From (40) we have

$$\Phi(f)=\int_0^{\pi/2}(s+1)f(s)\mathrm{d}s,\ \ F(f)=\int_0^{\pi}s^2f(s)\mathrm{d}s. \qquad (44)$$

It is evident that F, $\Phi\in C^*[0,\pi]$. Consequently we can take $Z=C[0,\pi]=X$. From (43), (44), (20), (21) we get

$$A_0^{-1}G_0=\int_0^t\cos s\,\mathrm{d}s-\frac{1}{2}\int_0^{\pi}\cos s\,\mathrm{d}s=\sin t,$$

$$\Phi\left(A_0^{-1}G_0\right)=\int_0^{\pi/2}(s+1)\sin s\,\mathrm{d}s=2,$$

$$A_0^{-1}S_0=-\cos t,$$

$$\Phi\left(A_0^{-1}S_0\right)=\int_0^{\pi/2}(s+1)(-\cos s)\,\mathrm{d}s=-\frac{\pi}{2},$$

$$\det L_0=\det\left[1-\Phi\left(A_0^{-1}G_0\right)\right]=1-2=-1\neq 0,\ \ L_0^{-1}=-1,$$

$$G=A_0^{-1}S_0+A_0^{-1}G_0L_0^{-1}\Phi\left(A_0^{-1}S_0\right)=\frac{\pi}{2}\sin t-\cos t,$$

$$F(G)=\int_0^{\pi}s^2\left(\frac{\pi}{2}\sin s-\cos s\right)\mathrm{d}s=\frac{\pi^3}{2},$$

then

$$\det L=\det\left[1-F(G)\right]=\frac{2-\pi^3}{2},\ \ L^{-1}=\frac{2}{2-\pi^3}.$$

Since $\det L\neq 0$, by Theorem 3 (ii), problem (36)–(39) or (34) is correct. Further by using (42) and taking into account that $A_0^{-1}G_0=\sin t$, we find

$$A^{-1}G=\cos t-\frac{\pi\sin t}{2}-\frac{\pi(2t-\pi)}{8},$$

$$A^{-1}A_0^{-1}G_0=\int_0^t(t-s)\sin s\,\mathrm{d}s+$$

$$+\frac{1}{4}\int_0^{\pi}\left(2s-3t-\frac{\pi}{2}\right)\sin s\,\mathrm{d}s=-\sin t-\frac{2t-\pi}{4}.$$

For $f(t)=\sin 2t$ by (42), (43) we calculate

$$A_0^{-1}f=\frac{1-\cos 2t}{2},\ \ \Phi\left(A_0^{-1}f\right)=(\pi+2)^2/16,$$

$$F\left(A_0^{-1}f\right)=\frac{\pi^3}{6}-\frac{\pi}{4},$$

$$A^{-1}A_0^{-1}f=\frac{1}{16}\left(4t^2-6\pi t+\pi^2-2+2\cos 2t\right).$$

Substituting these values into (13) we obtain the unique solution of (34), which is given by (35).

Example 3. Let $\Omega=\{(t,\ s)\in R\colon 0\le t,\ s\le 1$. The operator $B_1\colon C(\Omega)\to C(\Omega)$ corresponding to the problem

$$x''_{ts}(t,s)-t^3s\int_0^1\int_0^1 s^2x'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s-$$

$$-ts^2\int_0^1\int_0^1 tx'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s=5t^2+s,$$

$$x'_t,\ x''_{ts}\in C(\Omega),\ \ x(0,s)=s^2\int_0^1\int_0^1 x(t,s)\,\mathrm{d}t\,\mathrm{d}s,$$

$$x'_t(t,0)=t\int_0^1\int_0^1 sx'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s, \qquad (45)$$

is correct. The unique solution of problem (45) is given by the formula

$$x(t,s)=\frac{85148684t^2+31416680s^3t^2+287762400st^3}{172657440}+\frac{s^2\left(123613741+86328720t+15746840t^4\right)}{172657440}. \qquad (46)$$

Proof: If we compare (45) with (18), (19), it is natural to denote

$\mathbf{\Phi}=\Phi_1=\Phi$, $\mathbf{F}=F_1=F$, $\mathbf{G}_0=g_1^{(0)}=G_0$, $\mathbf{S}_0=s_1^{(0)}==S_0$, $\mathbf{I}_m=1$, and to take $X=C(\Omega)$,

$$B_1x(t)=x''_{ts}(t,s)-t^3s\int_0^1\int_0^1 s^2x'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s-$$

$$-ts^2\int_0^1\int_0^1 tx'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s=5t^2+s; \qquad (47)$$

$$D(B_1)=\{x(t,\ s)\in C(\Omega),\ x'_t,$$

$$x''_{ts}\in C(\Omega),\ x(0,\ s)=s^2\int_0^1\int_0^1 x(t,s)\,\mathrm{d}t\,\mathrm{d}s;$$

$$x'_t(t,0)=t\int_0^1\int_0^1 sx'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s\}; \qquad (48)$$

$$A_0Ax=x''_{ts}(t,s); \qquad (49)$$

$$Ax(t,s)=x'_t(t,s); \qquad (50)$$

$$D(A)=\{x(t,s)\in C(\Omega)\colon x'_t(t,s)\in C(\Omega),$$

$$x(0,s)=s^2\int_0^1\int_0^1 x(t,s)\,\mathrm{d}t\,\mathrm{d}s,$$

$$F(Ax)=\int_0^1\int_0^1 s^2x'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s,$$

$$\Phi(Ax)=\int_0^1\int_0^1 tx'_t(t,s)\,\mathrm{d}t\,\mathrm{d}s, \qquad (51)$$

$S_0=t^3s$, $G_0=ts^2$, $f=5t^2+s$. We denote $Ax(t,s)=x'_t(t,s)=y(t,s)=y$. Then from (48), (49) we have

$$y\in D(A_0),$$

$$A_0Ax=\left(x'_t(t,s)\right)'_s=y'_s(t,s)=A_0y(t,s),$$

$$y(t,0)=t\int_0^1\int_0^1 sy(t,s)\,\mathrm{d}t\,\mathrm{d}s.$$

So we proved that

$$A_0y=y'_s(t,s),$$

$$D(A_0)=\{y(t,s)\in C(\Omega)\colon y'_s\in C(\Omega),\ y(t,0)=$$

$$=t\int_0^1\int_0^1 sy(t,s)\,\mathrm{d}t\,\mathrm{d}s\}.$$

Now we check the condition $D(B_1)=D(AA_0)$. By definition

$$D(A_0A)=\{x(t,s)\in D(A): Ax(t,s)\in D(A_0)\}=$$
$$=\{x(t,s)\in C(\Omega): x'_t\in C(\Omega),\ x(0,s)=$$
$$=s^2\int_0^1\int_0^1 x(t,s)\mathrm{d}t\,\mathrm{d}s,\ x''_{ts}(t,s)\in C(\Omega),$$
$$x'_t(t,0)=t\int_0^1\int_0^1 sx'_t(t,s)\mathrm{d}t\,\mathrm{d}s\}=$$
$$=\{x(t,s)\in C(\Omega): x'_t,\ x''_{ts}\in C(\Omega),$$
$$x(0,s)=s^2\int_0^1\int_0^1 x(t,s)\mathrm{d}t\,\mathrm{d}s\},$$
$$x'_t(t,0)=t\int_0^1\int_0^1 sx'_t(t,s)\mathrm{d}t\,\mathrm{d}s\}=D(B_1).$$

So $D(B_1)=D(A_0A)$. It is easy to verify that the operators A, A_0 are correct on $C(\Omega)$ and for every $f(t,s)\in C(\Omega)$ the following hold true

$$A_0^{-1}f(t,s)=\int_0^s f(t,s_1)\mathrm{d}s_1+$$
$$+\frac{4t}{3}\int_0^1\int_0^1 s\int_0^s f(t,s_1)\mathrm{d}s_1\,\mathrm{d}t\,\mathrm{d}s; \quad (52)$$

$$A_0^{-1}f(t,s)=\int_0^t f(t_1,s)\mathrm{d}t_1+$$
$$+\frac{3s^2}{2}\int_0^1\int_0^1\int_0^t f(t_1,s)\mathrm{d}t_1\,\mathrm{d}t\,\mathrm{d}s. \quad (53)$$

From (51) for every $f(t,s)\in C(\Omega)$ we get

$$F(f)=\int_0^1\int_0^1 s^2 f(t,s)\mathrm{d}t\,\mathrm{d}s,$$
$$\Phi(f)=\int_0^1\int_0^1 tf(t,s)\mathrm{d}t\,\mathrm{d}s. \quad (54)$$

It is evident that $F,\Phi\in C^*(\Omega)$. Consequently we can take $Z=C(\Omega)=X$.

Further by using (52), (54), (20), (21) for $S_0=t^3s$, $G_0=ts^2$ we get

$$A_0^{-1}S_0=\int_0^s t^3s_1\,\mathrm{d}s_1+$$
$$+\frac{4t}{3}\int_0^1\int_0^1 s\int_0^s t^3s_1\,\mathrm{d}s_1\,\mathrm{d}t\,\mathrm{d}s=\frac{t}{24}+\frac{s^2t^3}{2},$$
$$A_0^{-1}G_0=\int_0^s ts_1^2\,\mathrm{d}s_1+$$
$$+\frac{4t}{3}\int_0^1\int_0^1 s\int_0^s ts_1^2\,\mathrm{d}s_1\,\mathrm{d}t\,\mathrm{d}s=\frac{2t}{45}+\frac{s^3t}{3},$$
$$F\left(A_0^{-1}G_0\right)=\int_0^1\int_0^1 s^2\left(\frac{2t}{45}+\frac{s^3t}{3}\right)\mathrm{d}t\,\mathrm{d}s=\frac{19}{540},$$
$$\Phi\left(A_0^{-1}G_0\right)=\int_0^1\int_0^1 t\left(\frac{2t}{45}+\frac{s^3t}{3}\right)\mathrm{d}t\,\mathrm{d}s=\frac{23}{540},$$
$$\det L_0=\det\left[1-\Phi\left(A_0^{-1}G_0\right)\right]=\frac{517}{540},\ L_0^{-1}=\frac{540}{517},$$

$$G=A_0^{-1}S_0+A_0^{-1}G_0L_0^{-1}\Phi\left(A_0^{-1}S_0\right)=$$
$$=\frac{t\left(907+340s^3+10340s^2t^2\right)}{20680},$$
$$F(G)=\frac{1393}{41360},\ \det L=\det\left[1-F(G)\right]=\frac{39967}{41360},$$
$$L^{-1}=\frac{41360}{39967}.$$

Since $\det L\neq 0$ then, by Theorem 3 (ii), problem (47), (48) or (45) is correct.

By (53) we calculate

$$A^{-1}G=\frac{907t^2+340s^3t^2+s^2\left(1013+5170t^4\right)}{41360},$$
$$A^{-1}A_0^{-1}G=\frac{120s^3t^2+23s^2+16t^2}{720}$$

and for $f(t,s)=5t^2+s$ by (52)–(54) we obtain

$$A_0^{-1}f=\frac{s^2}{2}+\frac{49t}{54}+5st^2,$$
$$A^{-1}A_0^{-1}f=s^2\left(\frac{t}{2}+\frac{287}{432}\right)+\frac{5st^3}{3}+\frac{49t^2}{108},$$
$$F\left(A_0^{-1}f\right)=\frac{541}{810},\ \Phi\left(A_0^{-1}f\right)=\frac{655}{648}.$$

Substituting these terms into (13) we obtain the unique solution of (45), which is given by (46).

Conclusion

The main research result of this paper is the existence and uniqueness of the operator equation $B_1u=f$ in the space setting of Banach spaces, given that $B_1=B_0B$. The necessary and sufficient conditions for the correctness of the operator B_1 are intermediate, secondary results. The solution procedure follows the universal decomposition method and provides a unique exact solution in closed form. This method can be also applied in more complex problems, as of the type $B_1u=f$, where $B_1=B_0B^2$ or $B_1=B_0^2B$ and for B_0, B given by (8), (9), respectively.

The entire approach is given in an algorithmic procedure that is reproducible in any program of symbolic calculations.

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Разложение абстрактных линейных операторов на банаховых пространствах

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Введение: большинство известных методов декомпозиции для решения краевых задач (метод декомпозиции Адомяна, естественное преобразование метода декомпозиции, модифицированный метод декомпозиции Адомяна, комбинированный метод преобразования Лапласа — декомпозиции Адомяна и метод декомпозиции области) используют так называемые полиномы Адомяна или итерации для получения приближенных решений. Насколько нам известно, прямой метод получения точного аналитического решения пока не предложен. **Цель:** разработать в произвольном банаховом пространстве новый универсальный метод разложения для класса обыкновенных интегро-дифференциальных уравнений или интегро-дифференциальных уравнений в частных производных с нелокальными и начальными граничными условиями в терминах абстрактного операторного уравнения $B_1 x = f$. **Результаты:** исследован класс интегро-дифференциальных уравнений в банаховом пространстве с нелокальными и начальными граничными условиями в терминах абстрактного операторного уравнения $B_1 x = \mathcal{A}x - \mathbf{S}_0\mathbf{F}(Ax) - \mathbf{G}_0\mathbf{\Phi}(Ax) = f$, $x \in D(B_1)$, где $\mathcal{A}$, A — линейные абстрактные операторы; $\mathbf{S}_0$, $\mathbf{G}_0$ — векторы, а $\mathbf{\Phi}$, $\mathbf{F}$ — функциональные векторы. Обычно $\mathcal{A}$, A — это линейные обыкновенные дифференциальные операторы или дифференциальные операторы в частных производных, а $\mathbf{F}(Ax)$, $\mathbf{\Phi}(Ax)$ — интегралы Фредгольма. Основным результатом нашего исследования является теорема существования и единственности уравнения $B_1x = f$ при условии, что оператор B_1 имеет разложение вида $B_1 = B_0B$, где B и B_0 — различные абстрактные линейные операторы специального вида. Предлагаемый метод разложения универсален и существенно отличается от других методов разложения в соответствующей литературе. Этот метод может быть применен как к обыкновенным интегро-дифференциальным уравнениям, так и к интегро-дифференциальным уравнениям в частных производных, и дает единственное точное решение в замкнутой аналитической форме в банаховом пространстве. Этапы метода решения иллюстрируются численными примерами, соответствующими конкретным задачам. Система компьютерной алгебры Mathematica используется для демонстрации результатов решения и оценки эффективности анализа. **Практическая значимость:** основным преимуществом настоящего метода решения является легкость его интеграции в интерфейс любого программного обеспечения CAS.

Ключевые слова — корректный оператор, разложение (факторизация, декомпозиция) операторов (уравнений), интегро-дифференциальные уравнения, краевые задачи, точное решение.

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